Dynamics of a dilute sheared inelastic fluid. II. The effect of correlations

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The effect of correlations on the viscosity of a dilute sheared inelastic fluid is analyzed using the ring-kinetic equation for the two-particle correlation function. The leading-order contribution to the stress in an expansion in $\epsilon=(1-e)^{1/2}$ is calculated, and it is shown that the leading-order viscosity is identical to that obtained from the Green-Kubo formula, provided the stress autocorrelation function in a sheared steady state is used in the Green-Kubo formula. A systemmatic extension of this to higher orders is also formulated, and the higher-order contributions to the stress from the ring-kinetic equation are determined in terms of the terms in the Chapman-Enskog solution for the Boltzmann equation. The series is resummed analytically to obtain a renormalized stress equation. The most dominant contributions to the two-particle correlation function are products of the eigenvectors of the conserved hydrodynamic modes of the two correlated particles. In Part I, it was shown that the long-time tails of the velocity autocorrelation function are not present in a sheared fluid. Using those results, we show that correlations do not cause a divergence in the transport coefficients; the viscosity is not divergent in two dimensions, and the Burnett coefficients are not divergent in three dimensions. The equations for three-particle and higher correlations are analyzed diagrammatically. It is found that the contributions due to the three-particle and higher correlation functions to the renormalized viscosity are smaller than those due to the two-particle distribution function in the limit $\epsilon\to 0$. This implies that the most dominant correlation effects are due to the two-particle correlations.

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I. INTRODUCTION

There are many practical applications which involve rapid granular flows under shear. In these flows, collisions between particles are inelastic and there is energy dissipation due to inelastic collisions. The source of energy for "fluidizing" the particles is provided by mean shear, and there is a balance between the rates of production and dissipation at steady state. Kinetic theories for flowing granular matter have been developed for some time now [1–7], using the analogy between the motion of the particles in the granular flow and the motion of molecules in a dilute gas. However, it has been realized that one of the limitations is the molecular chaos assumption in the formulation of the Boltzmann or Enskog equation for the single-particle distribution function. This is a significant limitation, since most practical applications involve flows in which the volume fraction is not small, and it is difficult to justify the application of the Boltzmann equation for dense flows. In order to develop confidence in the applicability of kinetic theory results for dense granular flows, it is necessary to examine the modification of the distribution function due to correlations. As a first step in this direction, we examine the effect of correlations in a dilute gas of inelastic particles under shear.

It has been realized for some time now that the constitutive relations obtained by solving the Boltzmann equation for a gas of elastic particles cannot be extended to dense gases, because correlations are neglected when we make the molecular chaos assumption in the Boltzmann equation. There have been several studies of the effect of correlations, notably by Kawasaki and Gunton [8] and Yamada and Kawasaki [9] using mode-coupling theory, Ernst and Dorfman [10], and Ernst *et al.* [11]. In addition, the Lutsko and Dufty [12] used a generalized Langevin formulation. All of these studies indicated that the shear viscosity in a fluid of elastic particles is a nonanalytic function of the strain rate. In two dimensions, the shear viscosity has the form $\mu=\mu_0+\mu'\ln(\dot{\gamma})$, while in three dimensions the shear viscosity has the form $\mu=\mu_0+\mu'|\dot{\gamma}|^{1/2}$, where μ_0 is the shear viscosity for a Newtonian fluid and $\dot{\gamma}$ is the strain rate. This implies that the coefficient of viscosity diverges in a two-dimensional fluid, while the Burnett coefficients diverge in a three-dimensional fluid. It is well known that the viscosity renormalization is caused by the long-time tails in the velocity autocorrelation functions [11,13], where the autocorrelation functions decay as a power law $t^{-D/2}$ in the long-time limit, where D is the dimension of the system.

The effect of correlations has been analyzed for a gas of inelastic particles in the homogeneous cooling state [14–16]. These studies consider an inelastic gas of particles in which the energy decreases with time as t^{-2} in the long-time limit due to viscous dissipation. However, it is possible to scale the temperature with a time-dependent scaling function, such that the scaled temperature is independent of time. The distribution of velocities is a Maxwell-Boltzmann distribution for velocities comparable to the square root of the scaled temperature, though there are high-velocity tails [17] which are not Gaussian. The system is usually prepared in such a way that the collisions are elastic up to $t=0$, and then inelasticity is switched on at $t=0$. In this case, the probability distribution function for the reference homogeneous cooling state is different from that for elastic fluids. From the known probability distribution function of the homogeneous cooling state, the relationship between the correlation and response functions is evaluated. It should be noted that the homogeneous cooling state is unstable to vorticity fluctuations [18], and therefore, the linear response calculation is carried out about an unstable base state.

In the present analysis, we study the effect of correlations on the transport coefficients for a sheared inelastic fluid by solving the ring-kinetic equation for a sheared granular flow, in a manner similar to Ernst *et al.* [11]. In this procedure, there is no necessity to assume a phase-space distribution for a reference state. The Boltzmann equation and the ring-kinetic equation are obtained by closure approximations in the Bogoliubov-Born-Green-Kirkwood-Yvon (BBKGY) hierarchy for the two-particle and three-particle distributions, respectively. In the ring-kinetic equation, the three-particle distribution function is written in terms of the two-particle distribution function in order to obtain closure at the two-particle level. In the calculation, the most important insight is that the dominant contribution to the two-particle distribution function is in the form of products of the conserved hydrodynamic modes of the colliding particles. Here, the conserved hydrodynamic modes are defined as the hydrodynamic modes of the linearized Boltzmann of Navier-Stokes equations for the sheared state for which the decay rate is zero in the long-wave (zero-wave-number) limit. For a gas of elastic particles, the conserved modes are linear combinations of mass, momentum, and energy, which are quantities conserved in an interparticle collision. For a gas of inelastic particles, the conserved variables are restricted to mass and momentum in the long-wave limit and energy is not conserved in the long-wave limit. Due to this crucial distinction, it turns out that our results are different from Ernst *et al.* [11], as discussed below.

It has been realized for some time now [19–25] that the hydrodynamic modes for a sheared inelastic fluid are very different from those for a fluid of elastic particles, because energy is not a conserved variable. Energy can be treated as a conserved variable for length scales $L \ll \lambda/\epsilon$ or wave number $k \gg \epsilon/\lambda$, where λ is the mean free path, $\epsilon=(1-e)^{1/2}$ is a small parameter which estimates the departure from elastic collisions, and e is the coefficient of restitution. For $L \gg \lambda/\epsilon$ or $k \ll \epsilon/\lambda$, the rate of dissipation of energy is large compared to the rate of conduction. Detailed expressions for the eigenvalues and eigenvectors of the hydrodynamic matrix were obtained in Part I. These results are used to determine the contribution to the viscosity due to two-particle correlations. The procedure used by Ernst *et al.* [11] involved solving a differential equation in wave vector space at steady state and is complicated by the necessity of prescribing boundary conditions in wave vector space. Here, we use a procedure similar to that of Lutsko and Dufty [12] for the solution of the Navier-Stokes equations with random forcing, where the unsteady equation is solved using an integration in time. For the case $k \gg \epsilon/\lambda$, where it is appropriate to treat energy as a conserved variable, the present procedure provides results identical to that of Ernst *et al.* [11]. For $k \ll (\epsilon/\lambda)$, we show that the results are very different. Since the linear hydrodynamic spectrum has decay rates proportional to $k^{2/3}$, there are no long-time tails in the decay of the velocity autocorrelation function [24]. As a result, the coefficient of viscosity is not singular in two dimensions and the Burnett coefficients are not singular in three dimensions.

An issue of interest is the relationship between the calculation of the viscosity via the ring-kinetic equation and the Green-Kubo relation. Lutsko and Dufty [12] showed that the results for the renormalized viscosity obtained from the Navier-Stokes equations with thermal noise are identical to those obtained by Ernst *et al.* [11] via the ring-kinetic equation. Whereas the analysis of Lutsko and Dufty [12] assumes a Gaussian random noise which satisfies the fluctuation dissipation theorem, there is no such assumption in the calculation of Ernst *et al.* [11]. Moreover, Lutsko and Dufty [12] obtained the viscosity from the energy balance equation, whereas Ernst *et al.* [11] directly calculated the streaming stress as the second moment of the fluctuating velocity in a dilute gas. This indicates that at a fundamental level, the results of the ring-kinetic equation are identical to those derived by fluctuating hydrodynamics. If this is so, then it should be demonstrable that the results of the ring-kinetic equation are identical to those of the Green-Kubo relation. We explore this in Sec. II and show that, in the leading approximation in the limit of small $\epsilon=(1-e_n)^{1/2}$, the ring-kinetic equation gives the same result as the Green-Kubo relation for the viscosity, provided the time correlation functions are calculated for the sheared steady state and not for the equilibrium state. An additional requirement is that the direction of shear should be reversed while calculating the time correlation function, since time-reversal symmetry is not preserved in a sheared system.

Though the ring-kinetic equation provides the same result as the Green-Kubo relation for the viscosity in the leading approximation, this approach is more powerful since it can be systematically extended to higher orders in the expansion. In addition, it can be used not only to find the shear stress, but also the normal stress differences and the mass and heat fluxes in a system with concentration and temperature gradients. This procedure is developed in Sec. II for the higher-order contributions to the stress in an $\epsilon=(1-e_n)^{1/2}$ expansion. We show that the solutions of the ring-kinetic equation can be systematically related to the solutions of the Boltzmann equation obtained using the Chapman-Enskog procedure, and the series can be analytically resummed to obtain a renormalized Dyson-type equation for the transport coefficients.

An agreement between the solutions of the ring-kinetic equation and the Green-Kubo relation has an additional subtle implication. Since the ring-kinetic equation includes only two-particle correlations, this agreement implies that the higher-order correlation functions do not influence the most singular part of the transport coefficients. This aspect is explored in Sec. III, where a diagrammatic perturbation expansion is used for the equations for the multiparticle correlation functions. It should be noted that the diagrammatic analysis used here is different from that in Dorfman and Cohen [26], where an expansion is carried out in different collision sequences. Here, the expansion is in correlation functions involving different numbers of particles. As noted earlier, the dominant contribution to the two-particle correlation function is in the form of products of the conserved hydrodynamic modes of the linearized Boltzmann or Navier-Stokes equations. We show that for the three-particle and higher-order correlation functions are small compared to the two-particle correlation function in the limit $\epsilon \to 0$; therefore, the dominant contribution to the viscosity arises due to the two-particle correlation function obtained by solving the ring-kinetic equation.

II. RING-KINETIC EQUATION

We consider a system of inelastic particles (disks or spheres) of diameter d subject to a linear shear flow. The flow is along the x direction, the velocity gradient is along the y direction, and the z coordinate is perpendicular to the plane of flow. The collision rules used here are those for smooth inelastic particles, where the post-collisional relative velocity along the line joining centers is $-e$ times the precollisional relative velocity, while the post-collisional relative velocity along perpendicular to the line joining centers is unchanged. Here, e is the normal coefficient of restitution. At steady state, there is a balance between the rate of production of energy due to mean shear, μG_{xy}^2, and the rate of dissipation of energy due to inelastic collisions, where G_{xy} is the strain rate. The rate of dissipation of energy is proportional to $\rho T^{3/2}(1-e^2)/\lambda$, since the dissipation of energy in a collision is proportional to $(1-e^2)T$ and the collision frequency is proportional to $T^{1/2}/\lambda$, where $T^{1/2}$ is the magnitude of the fluctuating velocity, $\lambda \sim 1/\rho d^2$ is the mean free path, and ρ is the number density when the mass of a particle is assumed to be 1. We consider the near-elastic limit $e \to 1$, where $\epsilon=(1-e)^{1/2}$ is a small parameter. In this case, it is easy to see that the strain rate is related to the temperature by $G_{xy} \sim \epsilon(T^{1/2}/\lambda)$. Since we scale all velocities in the analysis by $\bar{T}^{1/2}$ and all lengths by $1/\bar{\rho}d^2$, where $\bar{T}$ and $\bar{\rho}$ are the mean temperature and number density, respectively, we set $G_{xy}=\epsilon\dot{\gamma}$, where $\dot{\gamma}$ is $O(1)$ in the limit $\epsilon \to 0$.

Before proceeding, we briefly summarize the results of the linear analysis of Part I. The velocity scale in the present problem is $\sqrt{\bar{T}}$, where $\bar{T}$ is the mean temperature, and the length scale is the mean free path, $(\bar{\rho}d^2)^{-1}$, where $\bar{\rho}$ is the mean density. Here, the overbar is used to denote the mean values of the relevant variables in the base state. The velocity distribution function in the base state is obtained by solving the Boltzmann equation in the presence of imposed shear, and an asymptotic analysis is used in the parameter ϵ. The leading-order distribution function, denoted by F_α, is a Maxwell-Boltzmann distribution. The higher corrections can be systematically evaluated using the Chapman-Enskog procedure.

The dynamical equations for the mass, momentum, and energy are derived by taking the moments of the Boltzmann equation with the collisional invariants, which are denoted by the column vector $\Phi_\alpha=(1,(c_{\alpha x}/\sqrt{\bar{T}}),(c_{\alpha y}/\sqrt{\bar{T}}),(c_{\alpha z}/\sqrt{\bar{T}}),\sqrt{D/2}(c_\alpha^2/D\bar{T}-1))$, where D is the dimensionality of the system, and the Greek subscript α denotes a particle index. The particle fluctuating velocity is defined as $\mathbf{c}_\alpha=\mathbf{u}_\alpha-\mathbf{U}(\mathbf{x}_\alpha)$, where $\mathbf{U}(\mathbf{x}_\alpha)$ is the mean velocity at the particle position. For an elastic fluid, the collisional invariants are the particle mass, momentum, and energy; for a fluid of inelastic particles, energy is not a collisional invariant. The dispersion relations can be obtained in two ways. The first is to perturb the distribution function and then taking the moments of the linearized Boltzmann equation with respect to the collisional invariants. The other is to do a linear stability analysis of the macroscopic Navier-Stokes equations, which are obtained by taking moments of the Boltzmann equation. In the latter case, the macroscopic fields are considered to be of the form $(\bar{\rho}(1+\rho'),\epsilon\dot{\gamma}y+u_x',u_y',u_z',\sqrt{C_v}T')$. By inserting the macroscopic fields of this form into the Navier-Stokes equations and taking the Fourier transforms, we obtain the dispersion relation of the form

$$\partial_t\tilde{\phi}+\epsilon\dot{\gamma}k_y\frac{\partial}{\partial k_x}\tilde{\phi}+\tilde{L}\tilde{\phi}=0, \tag{1}$$

where $\tilde{L}$ is the dispersion matrix and the vector $\tilde{\phi}$ is

$$\tilde{\phi}=\left(\tilde{\rho},\left(\tilde{u}_x/\sqrt{\bar{T}}\right),\left(\tilde{u}_y/\sqrt{\bar{T}}\right),\left(\tilde{u}_z/\sqrt{\bar{T}}\right),\left(\tilde{T}/\bar{T}\right)\right), \tag{2}$$

where

$$\tilde{\star}(\mathbf{k})=\int d\mathbf{x}\,\exp(-\imath\mathbf{k}\cdot\mathbf{x})\star'(\mathbf{x}) \tag{3}$$

and $\tilde{L}$ is a matrix which depends on the wave vector. The derivative with respect to wave vector in Eq. (1) is inconvenient and can be removed using a time-dependent component k_y for the wave vector:

$$k_y(t)=k_y(0)-t\epsilon\dot{\gamma}k_x. \tag{4}$$

Expressed in terms of this wave vector, Eq. (1) reduces to the linear equation

$$\partial_t\tilde{\phi}+\tilde{L}\tilde{\phi}=0, \tag{5}$$

where the matrix $\tilde{L}$ is now time dependent due to Eq. (4). Due to this, both the eigenvalues and eigenvectors of $\tilde{L}$ are time dependent. It is not possible to obtain a closed-form solution for the eigenvectors of Eq. (5), but an iterative procedure has to be used, as discussed in Part I. In the following sections, we denote the eigenvalues of the matrix $\tilde{L}$ as $\lambda_I(\mathbf{k},t)$, where the capital Roman subscript is the index for the different eigenvalues. The matrix $\mathbf{E}$ is a matrix of dimension $N_c \times N_c$, whose columns are the eigenvectors, where N_c is the number of conserved variables. Using the eigenvalues and eigenfunctions, Eq. (5) can be reduced to

$$\partial_t\tilde{\Xi}+\Gamma\tilde{\Xi}=0, \tag{6}$$

where the matrix Γ is a diagonal matrix which contains the eigenvalues as its diagonal elements and the column vector Ξ is

$$\tilde{\Xi}=\mathbf{E}^{-1}\tilde{\phi}. \tag{7}$$

Note that the diagonal terms in the eigenvalue matrix $\mathbf{\Gamma}$ have dimensions of inverse time, while the eigenvector matrix $\mathbf{E}$ is dimensionless.

The nature of the hydrodynamic modes depends on the wavelength of the perturbations. For $k \gg \epsilon$, the rate of conduction of energy is large compared to the rate of dissipation, and it is appropriate to treat energy as a conserved variable. In this case, there are two propagating modes, two transverse shear modes, and one mode corresponding to energy diffusion, whose growth rates tend to zero in the long-wave limit. However, the diffusive modes are modified due to the time dependence of the wave vector in Eq. (4); in particular, in the long-time limit, they decay proportional to

$\exp(-\text{const}\times\dot{\gamma}^2k_x^2t^3/3)$, where const is the viscosity for the shear modes and the thermal diffusivity for the energy. The eigenvalues and eigenvectors of these modes were provided in Eqs. (32), (A7), and (A8) of Part I. For $k\ll\epsilon$, the rate of dissipation of energy is large compared to the rate of conduction. In this case, the growth rate of the fluctuations in the plane of shear is proportional to $k^{2/3}$ in the long wave limit, and the eigenvectors and eigenvalues were provided in Eqs. (44), (A10), and (A11) of Part I. It was also shown that the $k^{2/3}$ scaling is not valid for perturbations with wave vector in the gradient-vorticity plane; in this case, the scaling of the growth rate with wave vectors was similar to that for an elastic fluid, and the detailed expressions were provided in Eqs. (32) and (A7) of Part I. These eigenvalues and eigenvectors are used in the present analysis of the ring-kinetic equation.

For a steady shear flow, the equation for the single-particle distribution function reduces to

$$\left(\partial_t+\mathbf{c}_\alpha\cdot\frac{\partial}{\partial\mathbf{x}_\alpha}+\epsilon\dot{\gamma}y_\alpha\frac{\partial}{\partial x_\alpha}-\epsilon\dot{\gamma}c_{\alpha y}\frac{\partial}{\partial c_{\alpha x}}\right)f_\alpha=\int_\gamma C_{\alpha\gamma}[f_{\alpha\gamma}], \tag{8}$$

where

$$C_{\alpha\beta}(f_{\alpha\beta})=\int_{\mathbf{n}}[e^{-2}f_{\alpha\beta}(\mathbf{c}'_\alpha,\mathbf{c}'_\beta,-\mathbf{x}_{\alpha\beta})-f_{\alpha\beta}(\mathbf{c}_\alpha,\mathbf{c}_\beta,\mathbf{x}_{\alpha\beta})]\times(\mathbf{u}_\alpha-\mathbf{u}_\beta)\cdot\mathbf{n}d^2 \tag{9}$$

and the integral $\int_\gamma\equiv\int d\mathbf{c}_\gamma$. The one-particle and two-particle distribution functions are expanded in a series in the small parameter $\epsilon\sim d\dot{\gamma}/\bar{T}^{1/2}$:

$$f_\alpha=\bar{\rho}F_\alpha(1+g_\alpha), \tag{10}$$

$$f_{\alpha\beta}=\bar{\rho}^2F_\alpha F_\beta(1+g_\alpha+g_\beta+g_{\alpha\beta}). \tag{11}$$

The distribution F_α is the Chapman-Enskog solution for the Boltzmann equation:

$$-\epsilon\dot{\gamma}c_{\alpha y}\frac{\partial f_\alpha}{\partial c_{\alpha x}}=\int_\gamma C_{\alpha\gamma}[f_\alpha f_\gamma]. \tag{12}$$

If we insert the distributions (10) and (11) into the equation for the single-particle distribution function (8), we obtain the following relation between the single-particle distribution function:

$$(\partial_t+S_\alpha)[F_\alpha(1+g_\alpha)]=\int_\gamma C_{\alpha\gamma}[F_\alpha F_\gamma(1+g_\alpha+g_\gamma+g_{\alpha\gamma})], \tag{13}$$

where the modified "streaming" operator $S_\alpha(\psi)$ is defined as

$$S_\alpha(\psi)=\mathbf{c}_\alpha\cdot\frac{\partial\psi}{\partial\mathbf{x}_\alpha}+\epsilon\dot{\gamma}\left(y_\alpha\frac{\partial\psi}{\partial x_\alpha}\right)-\epsilon\dot{\gamma}\left(c_{\alpha y}\frac{\partial\psi}{\partial c_{\alpha x}}\right), \tag{14}$$

where ψ is some function of the velocities of particles α and β. In this and in future equations, we refer to $f_{\alpha\beta}$, $f_{\alpha\beta\gamma}$, etc., as the (two-particle, three-particle, etc.) "distribution functions," while g_α, $g_{\alpha\beta}$, etc., are referred to as the (single-particle, two-particle, etc.) "correlation functions."

The three-particle distribution function can be written in the most general form as

$$f_{\alpha\beta\gamma}=\bar{\rho}^3F_\alpha F_\beta F_\gamma(1+g_\alpha+g_\beta+g_\gamma+g_{\alpha\beta}+g_{\beta\gamma}+g_{\alpha\gamma}+g_{\alpha\beta\gamma}). \tag{15}$$

In the ring kinetic approximation, the term $g_{\alpha\beta\gamma}$ is neglected in Eq. (15), and the three-particle distribution function is written in terms of the one- and two-particle correlation functions. The equation for the two-body distribution is

$$\left(\frac{\partial}{\partial t}+S_\alpha+S_\beta\right)f_{\alpha\beta}=\delta(\mathbf{x}_{\alpha\beta})C_{\alpha\beta}[f_{\alpha\beta}]+\int_\gamma C_{\alpha\gamma}[f_{\alpha\beta\gamma}]+\int_\gamma C_{\beta\gamma}[f_{\alpha\beta\gamma}], \tag{16}$$

where the streaming operator S_α is defined in Eq. (14). The factor $\delta(\mathbf{x}_{\alpha\beta})$ on the first term in the right-hand side of the above equation is an approximation. For a collision between two particles α and β, the necessary condition is that the distance between the two particles is $(\mathbf{x}_\alpha-\mathbf{x}_\beta)=d\mathbf{a}$, where d is the particle diameter and $\mathbf{a}$ is the unit vector along the line joining the centers of the particles. Since we are interested in length scales large compared to the particle diameter, we have approximated $\mathbf{x}_{\alpha\beta}-d\mathbf{a}$ by $\mathbf{x}_{\alpha\beta}$ in Eq. (16).

Expressions (11) and (15) are used for $f_{\alpha\beta}$ and $f_{\alpha\beta\gamma}$ in the above equation; all spatial gradients of F_α and g_α are zero, since these are spatially invariant. With this simplification, and after using the equation for the one-particle distribution functions (12) and (13) to cancel some of the terms in the two-particle distribution function, we obtain

$$\begin{aligned}F_\alpha F_\beta&\left(\frac{\partial}{\partial t}+\epsilon\dot{\gamma}y_{\alpha\beta}\frac{\partial}{\partial x_{\alpha\beta}}+\mathbf{c}_\alpha\cdot\frac{\partial}{\partial\mathbf{x}_\alpha}+\mathbf{c}_\beta\cdot\frac{\partial}{\partial\mathbf{x}_\beta}+\epsilon\dot{\gamma}(S'_\alpha+S'_\beta)\right)g_{\alpha\beta}\\&=\bar{\rho}^{-1}\delta(\mathbf{x}_{\alpha\beta})C_{\alpha\beta}[F_\alpha F_\beta(1+g_\alpha+g_\beta+g_{\alpha\beta})]+\int_\gamma C_{\alpha\gamma}[F_\alpha F_\beta F_\gamma(g_{\alpha\beta}+g_{\beta\gamma})]+\int_\gamma C_{\beta\gamma}[F_\alpha F_\beta F_\gamma(g_{\alpha\gamma}+g_{\alpha\beta})],\end{aligned} \tag{17}$$

where $\mathbf{x}_{\alpha\beta}=\mathbf{x}_\alpha-\mathbf{x}_\beta$. There is a factor $\bar{\rho}^{-1}$ on the first term on the right-hand side of Eq. (17) because we have divided throughout by $\bar{\rho}^2(\bar{\rho}d^2\bar{T}^{1/2})\bar{T}^{-D}$ to nondimensionalize Eq. (17), where d is the particle diameter and $\bar{\rho}d^2\bar{T}^{1/2}$ is the inverse of a time scale, and the factor $\bar{T}^{-1}$ is due to the scaling of velocity by $\bar{T}^{1/2}$, and consequently the distribution function by $\bar{T}^{-D/2}$, where D is the dimensionality.

The modified streaming operators S'_α and S'_β in Eq. (17) incorporate the effect of the mean shear on the distribution functions F_α and F_β:

$$S'_\alpha = \left(- c_{\alpha y} \frac{\partial}{\partial c_{\alpha x}} + c_{\alpha x} c_{\alpha y} \right). \tag{18}$$

It is convenient to express the above equation in terms of the linearized Boltzmann operator L_α, defined as

$$L_\alpha(\psi) = \mathbf{c}_\alpha \frac{\partial}{\partial \mathbf{x}_\alpha} + \epsilon \dot{\gamma} S'_\alpha(\psi) - \int_\gamma C_{\alpha\gamma}[(1 + P_{\alpha\gamma})(F_\gamma \psi)], \tag{19}$$

where $P_{\alpha\gamma}$ is the "permutation operator" which changes the index α to γ. With this, the ring-kinetic equation in real space can be written as

$$F_\alpha F_\beta \left(\frac{\partial}{\partial t} + \epsilon \dot{\gamma} y_{\alpha\beta} \frac{\partial}{\partial x_{\alpha\beta}} + L_\alpha + L_\beta \right) g_{\alpha\beta} = \bar{\rho}^{-1} \delta(\mathbf{x}_{\alpha\beta}) C_{\alpha\beta}[F_\alpha F_\beta (1 + g_\alpha + g_\beta + g_{\alpha\beta})]. \tag{20}$$

III. SOLUTION OF THE RING-KINETIC EQUATION

A. Leading-order solution

The function g_α can be separated into two parts, $g_\alpha = h_\alpha + q_\alpha$. The first part h_α is the solution that would be obtained using the Chapman-Enskog procedure for the single-particle distribution function to leading order in a expansion in small h_α. At steady state in a homogeneous system, the distribution function is independent of time and position, and so the above equation simplifies to

$$\epsilon \dot{\gamma} F_\alpha S'_\alpha (1 + h_\alpha) = \int_\gamma C_{\alpha\gamma}[F_\alpha F_\gamma (1 + h_\alpha + h_\gamma + h_\alpha h_\gamma)]. \tag{21}$$

The other part q_α accounts for the two-particle correlations. Formally, it is necessary to write the equation for q_α as

$$\epsilon \dot{\gamma} F_\alpha S'_\alpha (q_\alpha) = \int_\gamma C_{\alpha\gamma}[F_\alpha F_\gamma (q_\alpha + q_\gamma + g_{\alpha\gamma})]. \tag{22}$$

However, in both Eqs. (21) and (22), we use a perturbation expansion $\epsilon = (1-e)^{1/2}$. As already noted, the strain rate is $(1-e)^{1/2}$ times $T^{1/2}/\lambda$, where T and λ are the temperature and mean free path. Therefore, it is possible to use an expansion of the above equations in which h_α and q_α are written formally as

$$h_\alpha = h_\alpha^{(1)} + h_\alpha^{(2)},$$

$$q_\alpha = q_\alpha^{(1)} + q_\alpha^{(2)},$$

$$g_{\alpha\gamma} = g_{\alpha\gamma}^{(1)} + g_{\alpha\gamma}^{(2)}. \tag{23}$$

The equation for the single-particle distribution function is best solved in real space, since the distribution function is independent of position and time in a homogeneous system. The equation for the leading order and the first correction to the single-particle distribution functions are

$$\int_\beta C_{\alpha\beta}[F_\alpha F_\beta] = 0, \tag{24}$$

$$\epsilon \dot{\gamma} F_\alpha S'_\alpha (1) = \int_\gamma C_{\alpha\gamma}[F_\alpha F_\gamma (1 + h_\alpha^{(1)} + h_\gamma^{(1)})], \tag{25}$$

$$\epsilon \dot{\gamma} F_\alpha S'_\alpha (h_\alpha^{(1)}) = \int_\gamma C_{\alpha\gamma}[F_\alpha F_\gamma (h_\alpha^{(2)} + h_\gamma^{(2)} + h_\alpha^{(1)} h_\gamma^{(1)})], \tag{26}$$

$$\int_\gamma C_{\alpha\gamma}[F_\alpha F_\gamma (q_\alpha^{(1)} + q_\gamma^{(1)} + g_{\alpha\gamma}^{(1)})] = 0, \tag{27}$$

$$\epsilon \dot{\gamma} F_\alpha S'_\alpha (q_\alpha^{(1)}) = \int_\gamma C_{\alpha\gamma}[F_\alpha F_\gamma (q_\alpha^{(2)} + q_\gamma^{(2)} + g_{\alpha\gamma}^{(2)})]. \tag{28}$$

Equations (24)–(26), are solved using a Sonine polynomial expansion to obtain the correction $h_\alpha^{(1)}$ and $h_\alpha^{(2)}$ in the Chapman-Enskog procedure [27], and this correction is used to obtain the viscous stress. It should be noted that $h_\alpha^{(1)}$ and $h_\alpha^{(2)}$ are $O(\epsilon)$ and $O(\epsilon^2)$ in an ϵ expansion, respectively. The first correction $h_\alpha^{(1)}$ is of the form

$$h_\alpha^{(1)} = \epsilon \dot{\gamma} c_{\alpha x} c_{\alpha y} G(c_\alpha), \tag{29}$$

where $G(c_\alpha)$ is only a function of the magnitude of the particle velocity. The next higher correction, which gives rise to the "Burnett" term in the equation for the stress tensor, has been calculated, though we do not provide the results explicitly here.

Equations (27) and (28) will be used, in the present analysis, to determine $q_\alpha^{(1)}$ and $q_\alpha^{(2)}$ in terms of $g_{\alpha\gamma}^{(1)}$ and $g_{\alpha\gamma}^{(2)}$, respectively, and this will then be used to calculate the stress due to correlations. Note that $h_\alpha^{(1)}$ is $O(\epsilon)$ from Eq. (25). However, the magnitude of $q_\alpha^{(1)}$ is to be determined from Eq. (27) by first solving for the two-particle correlation function $g_{\alpha\beta}^{(1)}$. It is known that $g_{\alpha\beta}^{(1)}$ turns out to be small compared to 1, but large compared to ϵ for an elastic gas [11].

To progress, ring-kinetic equation (20) is expressed in Fourier space using the Fourier transform for the two-particle correlation function:

$$\tilde{g}_{\alpha\beta}(\mathbf{k}, \mathbf{c}_\alpha, \mathbf{c}_\beta) = \int d\mathbf{x}_{\alpha\beta} g_{\alpha\beta}(\mathbf{x}_{\alpha\beta}, \mathbf{c}_\alpha, \mathbf{c}_\beta) \exp(-\imath \mathbf{k} \cdot \mathbf{x}_{\alpha\beta}). \tag{30}$$

The ring-kinetic equation can now be written as

$$F_\alpha F_\beta \left(\frac{\partial}{\partial t} + \epsilon \dot{\gamma} k_x \frac{\partial}{\partial k_y} + \tilde{L}_{\mathbf{k}}^\alpha + \tilde{L}_{-\mathbf{k}}^\beta \right) \tilde{g}_{\alpha\beta}(\mathbf{k}, t) = \bar{\rho}^{-1} C_{\alpha\beta}[F_\alpha F_\beta (h_\alpha + h_\beta + h_\alpha h_\beta + q_\alpha + q_\beta + g_{\alpha\beta})], \tag{31}$$

where $\tilde{L}_{\mathbf{k}}^\alpha$, the Fourier transform of the operator L_α, is

$$\tilde{L}^{\alpha}_{\mathbf{k}}(\psi) = \imath \mathbf{k}_{\alpha} \cdot \mathbf{c}_{\alpha} \psi - \epsilon \dot{\gamma} c_{\alpha x} \left(\frac{\partial \psi}{\partial c_{\alpha y}} - c_{\alpha y} \psi \right) - \int_{\xi} C_{\alpha \xi} [F_{\alpha} F_{\xi} (1 + P_{\alpha \xi}) \psi]. \tag{32}$$

Note that on the right-hand side of Eq. (31), it is necessary to use the values of h_{α}, h_{β}, q_{α}, q_{β}, and $g_{\alpha\beta}$ at contact; the only term that depends on $\mathbf{x}_{\alpha\beta}$ on the right-hand side of Eq. (20) is $\delta(\mathbf{x}_{\alpha\beta})$, and the Fourier transform of this is 1. Equation (31) can be expressed in terms of the expansions (23):

$$F_{\alpha} F_{\beta} \left(\frac{\partial}{\partial t} + \epsilon \dot{\gamma} k_x \frac{\partial}{\partial k_y} + \tilde{L}^{\alpha}_{\mathbf{k}} + \tilde{L}^{\beta}_{-\mathbf{k}} \right) \tilde{g}^{(1)}_{\alpha\beta}(\mathbf{k},t) = \bar{\rho}^{-1} C_{\alpha\beta} [F_{\alpha} F_{\beta} (h^{(1)}_{\alpha} + h^{(1)}_{\beta} + q^{(1)}_{\alpha} + q^{(1)}_{\beta} + g^{(1)}_{\alpha\beta})], \tag{33}$$

$$F_{\alpha} F_{\beta} \left(\frac{\partial}{\partial t} + \epsilon \dot{\gamma} k_x \frac{\partial}{\partial k_y} + \tilde{L}^{\alpha}_{\mathbf{k}} + \tilde{L}^{\beta}_{-\mathbf{k}} \right) \tilde{g}^{(2)}_{\alpha\beta}(\mathbf{k},t) = \bar{\rho}^{-1} C_{\alpha\beta} [F_{\alpha} F_{\beta} (h^{(2)}_{\alpha} + h^{(2)}_{\beta} + h^{(1)}_{\alpha} h^{(1)}_{\beta} + q^{(2)}_{\alpha} + q^{(2)}_{\beta} + g^{(2)}_{\alpha\beta})]. \tag{34}$$

We first examine the solutions for Eq. (33) for $g^{(1)}_{\alpha\beta}$. There are two possible ways to solve this equation. The first is to obtain a steady-state solution, in which the time derivative is set equal to zero. This is a differential equation in the wave vector $\mathbf{k}$, which can be explicitly solved if boundary conditions in the wave vector space are specified. The second option is to introduce a time-dependent wave vector which rotates with the shear flow, as was done in Eq. (4). Expressed in terms of the time-dependent wave vector, the differential equation reduces to

$$F_{\alpha} F_{\beta} \left(\frac{\partial}{\partial t} + \tilde{L}^{\alpha}_{\mathbf{k}} + \tilde{L}^{\beta}_{-\mathbf{k}} \right) \tilde{g}^{(1)}_{\alpha\beta} = \bar{\rho}^{-1} C_{\alpha\beta} [F_{\alpha} F_{\beta} (h^{(1)}_{\alpha} + h^{(1)}_{\beta} + q^{(1)}_{\alpha} + q^{(1)}_{\beta} + g^{(1)}_{\alpha\beta})]. \tag{35}$$

The above equation is an ordinary differential equation in time and can be solved subject to initial conditions. However, the wave vector in the above equation is now time dependent. We note, at this point, that the linearized Boltzmann operator $\tilde{L}^{\alpha}_{\mathbf{k}}$ is the same operator as that in Eq. (5), for which the eigenvalues λ_I were calculated in (32) and (44) in Part I and for which the eigenfunctions Ξ_I [Eq. (7)] were calculated in Eqs. (A7) and (A10) in Part I.

The eigenvalues λ_M and eigenfunctions $\xi_{\alpha M}$ of the operator $\tilde{L}_{\alpha}$, which have zero eigenvalues in the limit $\mathbf{k} \to 0$, satisfy the relation

$$\tilde{L}^{\alpha}_{\mathbf{k}}(\xi_{\alpha M}) = \lambda_M(\mathbf{k}) \xi_{\alpha M}, \tag{36}$$

where M varies from 1 to 5 for an elastic fluid which has five conserved modes and M varies from 1 to 4 for a sheared inelastic fluid with four conserved modes in three dimensions. Note that both λ_M and $\xi_{\alpha M}$ are, in general, functions of wave number and time due to the time dependence of the wave vector in Eq. (4). We also define the dual $\eta_{\alpha N}$ of the eigenfunctions $\xi_{\alpha M}$ by the orthogonality condition

$$\int_{\alpha} F_{\alpha} \xi_{\alpha M} \eta_{\alpha N} = \delta_{MN}. \tag{37}$$

For an elastic fluid, to leading order in a small-k expansion, the eigenvectors are orthonormal, and so $\eta_{\alpha M} = \xi_{\alpha M}$. This is not the case for an inelastic fluid.

The two-particle correlation function $\tilde{g}^{(1)}_{\alpha\beta}$ is written as an expansion in the eigenfunctions of the conserved modes:

$$\tilde{g}^{(1)}_{\alpha\beta}(\mathbf{k}(t),t) = A^{(1)}_{MN}(\mathbf{k}(t),t) \xi_{\alpha M}(\mathbf{k}(t),t) \xi_{\beta N}(-\mathbf{k}(t),t), \tag{38}$$

where a summation is implied due to the repeated capital Roman indices in the subscripts for the matrices. It should be noted that the eigenfunction $\xi_{\alpha M}$ is a linear combination of the eigenfunctions of the collision operator $\phi_{\alpha I}$, which are defined by the equation

$$C_{\alpha\beta}[(\phi_{\alpha I} + \phi_{\beta I})] = 0. \tag{39}$$

The eigenfunctions $\phi_{\alpha I}$ are the mass and the three components of the momentum and energy for an elastic system and include only the mass and the three components of the momentum for inelastic systems. These eigenfunctions are defined to be orthonormal:

$$\int_{\alpha} F_{\alpha} \Phi^{\dagger}_{\alpha} \Phi_{\alpha} = \mathbf{I}, \tag{40}$$

where $\mathbf{I}$ is the identity matrix and the superscript † denotes the transpose. The eigenfunctions $\xi_{\alpha M}$ can be defined in terms of the eigenfunctions of the collision operator as

$$\Xi_{\alpha} = \mathbf{E}^{-1} \Phi_{\alpha}, \tag{41}$$

where we use the bold characters Ξ_{α} and Φ_{α} to represent column vectors whose elements $\xi_{\alpha I}$ and $\phi_{\alpha I}$, and $\mathbf{E}$ is the matrix whose columns are the eigenvectors of the linear operator $\tilde{L}$ in Eq. (5). The dual Υ_{α} can be easily calculated from Eq. (41) and the orthogonality condition (40):

$$\Upsilon_{\alpha} = \mathbf{E}^{\dagger} \Phi_{\alpha}, \tag{42}$$

where the superscript † denotes the transpose. The column vectors $\eta_{\alpha I}$ in Eq. (37) are the columns of the matrix Υ_{α}.

Before inserting the expansion (38) into the repeated ring equation, we can make one more simplification to the right-hand side of Eq. (35). The eigenfunctions we are considering are linear combinations of those conserved in collisions, which satisfy the relation (39). Consider a function $\Psi(\mathbf{c}_{\alpha}, \mathbf{c}_{\beta}, \mathbf{k}(t))$ which is a function of the velocities of the colliding particles. We write the integral $\mathcal{W}_{IJ}$ as

$$\mathcal{W}_{IJ} = \int_{\alpha} \int_{\beta} \phi_{\alpha I} \phi_{\beta J} C_{\alpha\beta} [F_{\alpha} F_{\beta} \Psi]. \tag{43}$$

Due to the symmetry of the collision operator, the above can be rewritten as

$$\mathcal{W}_{IJ} = \int_\alpha \int_\beta F_\alpha F_\beta \Psi C_{\alpha\beta}[\phi_{\alpha I}\phi_{\beta J}]$$

$$= \frac{1}{2}\int_\alpha \int_\beta F_\alpha F_\beta \Psi C_{\alpha\beta}[\phi_{\alpha I}\phi_{\beta J} + \phi_{\beta I}\phi_{\alpha J}]. \quad (44)$$

The last step follows from the symmetry of $\mathcal{W}_{IJ}$ under the interchange of α and β. The above integral is further simplified as follows. We first note that since $\psi_{\alpha I}$ and $\psi_{\beta J}$ are collisional invariants:

$$C_{\alpha\beta}[(\phi_{\alpha I} + \phi_{\beta I})(\phi_{\alpha J} + \phi_{\beta J})] = 0. \quad (45)$$

Therefore, the right-hand side of Eq. (44) can be rewritten as

$$\mathcal{W}_{IJ} = -\frac{1}{2}\int_\alpha \int_\beta F_\alpha F_\beta \Psi C_{\alpha\beta}[\phi_{\alpha I}\phi_{\alpha J} + \phi_{\beta I}\phi_{\beta J}]$$

$$= -\int_\alpha \int_\beta F_\alpha F_\beta \Psi C_{\alpha\beta}[\phi_{\alpha I}\phi_{\alpha J}]. \quad (46)$$

The last step in Eq. (46) once again follows from the symmetry of $\mathcal{W}_{IJ}$ with respect to an interchange in the particle identities α and β. The symmetry of the collision operator can once again be used to write

$$\mathcal{W}_{IJ} = -\int_\alpha \int_\beta \phi_{\alpha I}\phi_{\alpha J} C_{\alpha\beta}[F_\alpha F_\beta \Psi]$$

$$= -\int_\alpha \phi_{\alpha I}\phi_{\alpha J}\int_\beta C_{\alpha\beta}[F_\alpha F_\beta \Psi]. \quad (47)$$

For the right-hand side of Eq. (35), the transformation effected from Eq. (43)–(47) can be written as

$$\int_\alpha \int_\beta \phi_{\alpha I}\phi_{\beta J} C_{\alpha\beta}[F_\alpha F_\beta (h_\alpha^{(1)} + h_\beta^{(1)} + q_\alpha^{(1)} + q_\beta^{(1)} + g_{\alpha\beta}^{(1)})]$$

$$= -\int_\alpha \phi_{\alpha I}\phi_{\alpha J}\int_\beta C_{\alpha\beta}[F_\alpha F_\beta (h_\alpha^{(1)} + h_\beta^{(1)} + q_\alpha^{(1)} + q_\beta^{(1)} + g_{\alpha\beta}^{(1)})]$$

$$= -\epsilon\dot\gamma \int_\alpha \phi_{\alpha I}\phi_{\alpha J} F_\alpha S'_\alpha(1), \quad (48)$$

where the final step results from the equalities (25) and (27), and the operator S'_α is given in Eq. (18). This reduction shows that if we are only interested in the projections of the ring equation (35) onto the conserved modes of the collision operator, it is sufficient to consider the "simple ring" equation

$$F_\alpha F_\beta \left(\frac{\partial}{\partial t} + \tilde{L}_{\mathbf{k}}^\alpha + \tilde{L}_{-\mathbf{k}}^\beta\right)\tilde{g}_{\alpha\beta}^{(1)} = \bar\rho^{-1} C_{\alpha\beta}[F_\alpha F_\beta (h_\alpha^{(1)} + h_\beta^{(1)})]. \quad (49)$$

The simple ring equation can now be solved explicitly, since the inhomogeneous term on the right-hand side of the equation is known from the Chapman-Enskog equation (24). The expansion (38) is inserted into the simple ring equation (49) and multiplied by $\eta_{\alpha P}(\mathbf{k}(t))\eta_{\beta Q}(-\mathbf{k}(t))$ and integrated over the velocities $\mathbf{c}_\alpha$ and $\mathbf{c}_\beta$, to obtain

$$\left(\frac{\partial}{\partial t} + \lambda_P(\mathbf{k}(t)) + \lambda_Q(-\mathbf{k}(t))\right)A_{PQ}^{(1)}(\mathbf{k}(t),t) = R_{PQ}^{(1)}(\mathbf{k}(t),t), \quad (50)$$

where the inhomogeneous term $R_{PQ}^{(1)}$ on the right-hand side is

$$R_{PQ}^{(1)}(\mathbf{k}(t),t) = \bar\rho^{-1}\epsilon\dot\gamma \int_\alpha \int_\beta \eta_{\alpha P}(\mathbf{k}(t),t)\eta_{\beta Q}(-\mathbf{k}(t),t)$$
$$\times C_{\alpha\beta}[F_\alpha F_\beta (h_\alpha^{(1)} + h_\beta^{(1)} + q_\alpha^{(1)} + q_\beta^{(1)} + g_{\alpha\beta}^{(1)})]$$
$$= -\bar\rho^{-1}\epsilon\dot\gamma \int_\alpha \eta_{\alpha P}(\mathbf{k}(t),t)\eta_{\alpha Q}(-\mathbf{k}(t),t)$$
$$\times \int_\beta C_{\alpha\beta}[F_\alpha F_\beta (h_\alpha^{(1)} + h_\beta^{(1)} + q_\alpha^{(1)} + q_\beta^{(1)} + g_{\alpha\beta}^{(1)})]$$
$$= -\bar\rho^{-1}\epsilon\dot\gamma \int_\alpha \eta_{\alpha P}(\mathbf{k}(t),t)\eta_{\alpha Q}(-\mathbf{k}(t),t)F_\alpha S'_\alpha(1). \quad (51)$$

Note that both $R_{PQ}^{(1)}$ and $A_{PQ}^{(1)}$ are functions of time due to the time dependence of the wave vector, and they also have explicit time dependence because the duals $\eta_{\alpha P}$ have explicit time dependence. The solution of Eq. (50) is of the form

$$A_{PQ}^{(1)}(\mathbf{k}(t),t) = \int_{-\infty}^t dt' \exp\left(-\int_{t'}^t dt''[\lambda_P(\mathbf{k}(t'')) + \lambda_Q(-\mathbf{k}(t''))]\right)R_{PQ}^{(1)}(\mathbf{k}(t'),t'). \quad (52)$$

For a steady homogeneous flow with strain rate $\epsilon\dot\gamma\mathbf{e}_x\mathbf{e}_y$, the inhomogeneous term R_{PQ} on the right-hand side is

$$R_{PQ}^{(1)}(\mathbf{k}(t),t) = -\bar\rho^{-1}\epsilon\dot\gamma E_{PI}^\dagger(\mathbf{k}(t),t)E_{QJ}^\dagger(-\mathbf{k}(t),t)$$
$$\times \int_\alpha \phi_{\alpha I}\phi_{\alpha J}F_\alpha S'_\alpha(1)$$
$$= -\bar\rho^{-1}\epsilon\dot\gamma E_{PI}^\dagger(\mathbf{k}(t),t)E_{QJ}^\dagger(-\mathbf{k}(t),t)\Gamma_{IJ}, \quad (53)$$

where the matrix Γ_{IJ} is

$$\Gamma_{IJ} = \int_\alpha F_\alpha \phi_{\alpha I}\phi_{\beta J}S'_\alpha(1) = \int_\alpha F_\alpha \phi_{\alpha I}\phi_{\beta J}(c_{\alpha x}c_{\alpha y}/\bar T). \quad (54)$$

Using this, the final result for the coefficients A_{PQ} can be factored into two parts, one of which depends on wave number and time and the other of which is only a function of the particle velocities:

$$A_{PQ}^{(1)}(\mathbf{k},t) = -\bar{\rho}^{-1}\epsilon\dot{\gamma}\Gamma_{IJ}\left[\int_{-\infty}^{t} dt' \exp\left(-\int_{t'}^{t} dt''[\lambda_P(\mathbf{k}(t'')) + \lambda_Q(-\mathbf{k}(t''))]\right) E_{PI}^{\dagger}(\mathbf{k}(t'),t')E_{QJ}^{\dagger}(-\mathbf{k}(t'),t')\right], \tag{55}$$

where the matrix Γ_{IJ} is given in Eq. (54). Note that the all wave vectors within the integrals in the right-hand side of Eq. (55) are dependent on time, as given in Eq. (4).

The solution for $q_\alpha^{(1)}$ has to be evaluated from Eq. (27), using the solutions (38) and (52) for $\tilde{g}_{\alpha\beta}^{(1)}$. In Eq. (27), the value of $g_{\alpha\beta}^{(1)}$ within the integral is at collision, when the distance between particles is equal to the particle diameter. Since we are considering the dilute limit where this distance is zero, the value of $g_{\alpha\beta}$ to be inserted into Eq. (27) is

$$\begin{aligned} g_{\alpha\beta}^{(1)}(\mathbf{x}_{\alpha\beta}=0) &= \int_{\mathbf{k}} \tilde{g}_{\alpha\beta}^{(1)} \\ &= \int_{\mathbf{k}} A_{PQ}^{(1)}(\mathbf{k}(t),t)\xi_{\alpha P}(\mathbf{k}(t),t)\xi_{\beta Q}(-\mathbf{k}(t),t) \\ &= \phi_{\alpha I}\phi_{\beta J}\int_{\mathbf{k}} A_{PQ}^{(1)}(\mathbf{k}(t),t)E_{PI}^{-1}(\mathbf{k}(t),t)E_{QJ}^{-1}(-\mathbf{k}(t),t). \end{aligned} \tag{56}$$

When this is inserted into Eq. (27), we obtain,

$$\int_\beta C_{\alpha\beta}[F_\alpha F_\beta(q_\alpha^{(1)} + q_\beta^{(1)})] + \int_{\mathbf{k}} A_{PQ}^{(1)}(\mathbf{k}(t),t)E_{PI}^{-1}(\mathbf{k}(t)) \times E_{QJ}^{-1}(-\mathbf{k}(t))\int_\beta C_{\alpha\beta}[F_\alpha F_\beta \phi_{\alpha I}\phi_{\beta J}] = 0. \tag{57}$$

The above equation can be solved by expanding $q_\alpha^{(1)}$ in a set of orthogonal functions $H_{\alpha K}(\mathbf{c}_\alpha)$ and then finding the coefficients in the expansion from the above equation. The last term on the right-hand side of Eq. (57) can be transformed as

$$\begin{aligned} &\int_\alpha H_{\alpha K}\int_\beta C_{\alpha\beta}[F_\alpha F_\beta \phi_{\alpha I}\phi_{\beta J}] \\ &= \frac{1}{2}\int_\alpha\int_\beta (H_{\alpha K} + H_{\beta K})C_{\alpha\beta}[F_\alpha F_\beta \phi_{\alpha I}\phi_{\beta J}] \\ &= \frac{1}{4}\int_\alpha\int_\beta (H_{\alpha K} + H_{\beta K})C_{\alpha\beta}[F_\alpha F_\beta(\phi_{\alpha I}\phi_{\beta J} + \phi_{\alpha J}\phi_{\beta I})] \\ &= -\frac{1}{4}\int_\alpha\int_\beta (H_{\alpha K} + H_{\beta K})C_{\alpha\beta}[F_\alpha F_\beta(\phi_{\alpha I}\phi_{\alpha J} + \phi_{\beta I}\phi_{\beta J})] \\ &= -\frac{1}{4}\int_\alpha\int_\beta F_\alpha F_\beta(\phi_{\alpha I}\phi_{\alpha J} + \phi_{\beta I}\phi_{\beta J})]C_{\alpha\beta}[H_{\alpha K} + H_{\beta K}]. \end{aligned} \tag{58}$$

In going from the third to fourth step above, we have used the result that if $\phi_{\alpha I}$ and $\phi_{\beta J}$ are collisional invariants, then $C_{\alpha\beta}[F_\alpha F_\beta(\phi_{\alpha I} + \phi_{\alpha J})(\phi_{\beta I} + \phi_{\beta J})] = 0$. From the first two terms on the left-hand side of Eq. (58), when multiplied by $H_{\alpha K}$ and integrated over all velocities, we find

$$\begin{aligned} &\int_\alpha H_{\alpha K}\int_\beta C_{\alpha\beta}[F_\alpha F_\beta(q_\alpha^{(1)} + q_\beta^{(1)})] \\ &= \frac{1}{2}\int_\alpha\int_\beta (H_{\alpha K} + H_{\beta K})C_{\alpha\beta}[F_\alpha F_\beta(q_\alpha^{(1)} + q_\beta^{(1)})] \\ &= -\frac{1}{2}\int_\alpha\int_\beta F_\alpha F_\beta(q_\alpha^{(1)} + q_\beta^{(1)})C_{\alpha\beta}[H_{\alpha K} + H_{\beta K}]. \end{aligned} \tag{59}$$

Comparing Eqs. (58) and (59), it is clear that the solution for $q_\alpha^{(1)}$ is

$$\begin{aligned} q_\alpha^{(1)} &= -\phi_{\alpha I}\phi_{\alpha J}\left(\frac{1}{2}\int_{\mathbf{k}} A_{PQ}^{(1)}(\mathbf{k}(t),t)E_{PI}^{-1}(\mathbf{k}(t),t)E_{QJ}^{-1}(-\mathbf{k}(t),t)\right) \\ &= \phi_{\alpha I}\phi_{\alpha J}K_{IJ}^{LM}\left(\int_\beta \phi_{\beta L}\phi_{\beta M}F_\beta S_\beta'(1)\right), \end{aligned} \tag{60}$$

where

$$\begin{aligned} K_{IJ}^{LM} = \Bigg[&\frac{\bar{\rho}^{-1}\epsilon\dot{\gamma}}{2}\int_{-\infty}^{t} dt'\int_{\mathbf{k}} E_{PI}^{-1}(\mathbf{k}(t),t)E_{QJ}^{-1}(-\mathbf{k}(t),t) \\ &\times\exp\left(-\int_{t'}^{t} dt''[\lambda_P(\mathbf{k}(t'')) + \lambda_Q(-\mathbf{k}(t''))]\right) \\ &\times E_{PL}^{\dagger}(\mathbf{k}'(t'),t')E_{QM}^{\dagger}(-\mathbf{k}'(t'),t')\Bigg]. \end{aligned} \tag{61}$$

Because we are integrating over all wave vectors in the above expression and due to time-translational invariance, it is clear that K_{IJ}^{MN} is independent of position and time.

The stress due to the first correction $q_\alpha^{(1)}$ is evaluated as

$$\begin{aligned} \sigma_{xy}^{(1)} &= -\bar{\rho}\int_\alpha F_\alpha c_{\alpha x}c_{\alpha y}q_\alpha^{(1)} \\ &= -\bar{\rho}\left(\int_\alpha F_\alpha c_{\alpha x}c_{\alpha y}\phi_{\alpha I}\phi_{\alpha J}\right) \\ &\quad\times K_{IJ}^{LM}\left(\int_\alpha F_\alpha \phi_{\alpha L}\phi_{\alpha M}S_\alpha'(1)\right). \end{aligned} \tag{62}$$

Here, the fourth term on the right-hand side is given by Eq. (54), while the second term is also obtained from Eq. (54) as

$$\int_\alpha F_\alpha c_{\alpha x}c_{\alpha y}\phi_{\alpha I}\phi_{\alpha J} = \Gamma_{IJ}\bar{T}. \tag{63}$$

For comparison with the Green-Kubo formula, we also provide the following equivalent expression for the stress tensor:

$$\sigma_{xy}^{(1)} = \frac{\epsilon\dot{\gamma}\bar{T}}{2}\Gamma_{IJ}\Gamma_{KL}\int_{\mathbf{k}} E_{PI}^{-1}(\mathbf{k}(t),t)E_{QJ}^{-1}(-\mathbf{k}(t),t) \times\left[\int_{-\infty}^{t} dt' \exp\left(-\int_{t'}^{0} dt''[\lambda_P(\mathbf{k}(t'')) + \lambda_Q(-\mathbf{k}(t''))]\right)E_{PK}^{\dagger}(\mathbf{k}(t'),t')E_{QL}^{\dagger}(-\mathbf{k}(t'),t')\right]. \tag{64}$$

It is useful to check the dimensions of the stress in Eq. (64). We have assumed that the mass of a particle is equal to 1 without loss of generality. The dimensions of the mean temperature $\bar{T}$ is L^2T^{-2}, where L and T and the length and time dimensions. The eigenvectors are defined to be dimensionless, while the integral over the wave vector has dimensions of L^{-3}. Therefore, the stress has dimensions of $L^{-1}T^{-2}$, as expected. The leading contribution to the viscosity is obtained by taking the ratio of the stress and strain rate $(-\sigma_{xy}^{(1)}/\epsilon\dot{\gamma})$:

$$\mu = \frac{\bar{T}}{2}\Gamma_{IJ}\Gamma_{KL}\int_{\mathbf{k}} E_{PI}^{-1}(\mathbf{k}(t),t)E_{QJ}^{-1}(-\mathbf{k}(t),t) \times\left[\int_{-\infty}^{t} dt' \exp\left(-\int_{t'}^{0} dt''[\lambda_P(\mathbf{k}(t'')) + \lambda_Q(-\mathbf{k}(t''))]\right) \times E_{PK}^{\dagger}(\mathbf{k}(t'),t')E_{QL}^{\dagger}(-\mathbf{k}(t'),t')\right]. \tag{65}$$

B. Higher corrections

The solution for the first correction can be determined from Eqs. (26), (28), and (34). The method of solution is discussed only briefly, since the solution procedures are identical to those for the leading order solution. The first correction to the two-particle correlation function is written, equivalent to Eq. (38), as

$$\tilde{g}_{\alpha\beta}^{(2)} = A_{MN}^{(2)}(\mathbf{k}(t),t)\xi_{\alpha M}(\mathbf{k}(t),t)\xi_{\beta N}(-\mathbf{k}(t),t). \tag{66}$$

When this is inserted into Eq. (35), and we obtain the equivalent of Eq. (50):

$$\left(\frac{\partial}{\partial t} + \lambda_P(\mathbf{k}(t)) + \lambda_Q(-\mathbf{k}(t))\right)A_{PQ}^{(2)}(\mathbf{k}(t),t) = R_{PQ}^{(2)}(\mathbf{k}(t),t), \tag{67}$$

where

$$\begin{aligned} R_{PQ}^{(2)}(\mathbf{k}(t),t) &= -\bar{\rho}^{-1}\epsilon\dot{\gamma}\int_{\alpha}\eta_{\alpha P}(\mathbf{k}(t),t)\eta_{\alpha Q}(-\mathbf{k}(t),t)F_{\alpha}S'_{\alpha}(h_{\alpha}^{(1)} + q_{\alpha}^{(1)}) \\ &= -\bar{\rho}^{-1}\epsilon\dot{\gamma}\int_{\alpha}E_{PI}^{\dagger}(\mathbf{k}(t),t)E_{QJ}^{\dagger}(-\mathbf{k}(t),t)\phi_I\phi_J F_{\alpha}S'_{\alpha}(h_{\alpha}^{(1)} + q_{\alpha}^{(1)}). \end{aligned} \tag{68}$$

The term $S'_{\alpha}(q_{\alpha}^{(1)})$ can be simplified, using Eq. (60), as

$$F_{\alpha}S'_{\alpha}(q_{\alpha}^{(1)}) = F_{\alpha}S'_{\alpha}(\phi_{\alpha I}\phi_{\alpha J})K_{IJ}^{MN}\int_{\beta}F_{\beta}\phi_{\beta M}\phi_{\beta N}S'_{\beta}(1). \tag{69}$$

Therefore, the solution for $R_{PQ}^{(2)}$ assumes the form

$$\begin{aligned} R_{PQ}^{(2)}(\mathbf{k}(t),t) &= -\bar{\rho}^{-1}\epsilon\dot{\gamma}E_{PI}^{\dagger}(\mathbf{k}(t),t)E_{QJ}^{\dagger}(-\mathbf{k}(t),t) \times\left(\int_{\alpha}\phi_{\alpha I}\phi_{\alpha J}F_{\alpha}S'_{\alpha}(h_{\alpha}^{(1)})\right) \\ &= -\bar{\rho}^{-1}\epsilon\dot{\gamma}E_{PI}^{\dagger}(\mathbf{k}(t),t)E_{QJ}^{\dagger}(-\mathbf{k}(t),t) \times\Theta_{IJ}^{KL}K_{KL}^{MN}\int_{\beta}F_{\beta}\phi_{\beta M}\phi_{\beta N}S'_{\beta}(1), \end{aligned} \tag{70}$$

where K_{KL}^{MN} is given in Eq. (61) and

$$\Theta_{IJ}^{KL} = \left(\int_{\alpha}F_{\alpha}\phi_I\phi_J S'_{\alpha}(\phi_K\phi_L)\right). \tag{71}$$

It is useful to discuss, in a little further detail, the formulation of the matrices K_{KL}^{MN} and Θ_{IJ}^{KL}. If N_c is the number of conserved variables, then each of these matrices has N_c^4 elements, since each index goes from 1 to N_c. This can be framed into a two-dimensional square matrix with N_c^2 rows and columns. In the matrix K_{KL}^{MN}, for example, the column index is assigned in terms of the subscripts as $(M-1)N_C+N$, while the row index is assigned in terms of the superscripts as $(K-1)N_c+L$. The product $\Theta_{IJ}^{KL}K_{KL}^{MN}$ is then the usual matrix multiplication of the two-dimensional matrix, where the rows of Θ_{IJ}^{KL} are multiplied by the columns of the K_{KL}^{MN}. In this formulation, the last term on the right-hand side of Eq. (74) below, $\int_{\beta}F_{\beta}\phi_{\beta M}\phi_{\beta N}S'_{\beta}(1)$ is a column matrix of dimension $1\times N_c^2$, where the row index is given by $(M-1)N_c+N$, and the first term on the right-hand side of Eq. (74) below, $\int_{\alpha}F_{\alpha}c_{\alpha x}c_{\alpha y}\phi_{\alpha I}\phi_{\alpha J}$, is a row matrix of dimension $N_c\times 1$, where the row index is given by $(I-1)N_c+J$. The matrix $\mathcal{I}_{IJ}^{MN}$ in Eq. (76) below is an identity matrix of dimension $N_c^2\times N_c^2$ in this formulation.

The solution for $A_{PQ}^{(2)}(\mathbf{k},t)$, obtained from solving Eq. (67), is

$$A_{PQ}^{(2)} = -\int_{-\infty}^{t} dt' E_{PI}^{\dagger}(\mathbf{k}(t',t'),t') E_{QJ}^{\dagger}(-\mathbf{k}(t'),t') \times \exp\left(-\int_{t'}^{t} dt''[\lambda_P(\mathbf{k}(t'')) + \lambda_Q(-\mathbf{k}(t''))]\right) \times \left(\int_{\alpha} F_\alpha \phi_{\alpha I}\phi_{\alpha J} S'_\alpha(h_\alpha^{(1)})\right)$$

$$= -\int_{-\infty}^{t} dt' E_{PI}^{\dagger}(\mathbf{k}(t',t'),t') E_{QJ}^{\dagger}(-\mathbf{k}(t'),t') \times \exp\left(-\int_{t'}^{t} dt''[\lambda_P(\mathbf{k}(t'')) + \lambda_Q(-\mathbf{k}(t''))]\right) \times \Theta_{KL}^{IJ} K_{KL}^{MN} \int_{\beta} F_\beta \phi_{\beta M}\phi_{\beta N} S'_\beta(1). \quad (72)$$

From this, the solution for $q_\alpha^{(2)}$, equivalent to (60), is

$$q_\alpha^{(2)} = \phi_{\alpha A}\phi_{\alpha B} K_{AB}^{IJ}\left(\int_{\alpha} F_\alpha \phi_{\alpha I}\phi_{\alpha J} S'_\alpha(h_\alpha^{(1)})\right) + \phi_A \phi_B K_{AB}^{IJ} \Theta_{IJ}^{KL} K_{KL}^{MN}\left(\int_{\beta} F_\beta \phi_{\beta M}\phi_{\beta N} S'_\beta(1)\right). \quad (73)$$

The above calculation can easily be extended to the obtain the second correction to the stress as

$$\sigma_{xy}^{(2)} = -\bar{\rho}\left(\int_{\alpha} F_\alpha c_{\alpha x} c_{\alpha y}\phi_{\alpha I}\phi_{\alpha J}\right) K_{IJ}^{LM}\left(\int_{\alpha} F_\alpha \phi_{\alpha L}\phi_{\alpha M} S'_\alpha(h_\alpha^{(1)})\right)$$

$$= -\bar{\rho}\left(\int_{\alpha} F_\alpha c_{\alpha x} c_{\alpha y}\phi_{\alpha A}\phi_{\alpha B}\right) K_{AB}^{IJ}\Theta_{IJ}^{KL} K_{KL}^{MN} \times \left(\int_{\beta} F_\beta \phi_{\beta M} S'_\beta(1)\right). \quad (74)$$

Higher corrections can be obtained in similar manner a similar to Eqs. (62) and (74). For example, the third correction is

$$\sigma_{xy}^{(3)} = -\bar{\rho}\left(\int_{\alpha} F_\alpha c_{\alpha x} c_{\alpha y}\phi_{\alpha I}\phi_{\alpha J}\right) K_{IJ}^{LM}\left(\int_{\alpha} F_\alpha \phi_{\alpha L}\phi_{\alpha M} S'_\alpha(h_\alpha^{(2)})\right)$$

$$= -\bar{\rho}\left(\int_{\alpha} F_\alpha c_{\alpha x} c_{\alpha y}\phi_{\alpha A}\phi_{\alpha B}\right) K_{AB}^{IJ}\Theta_{IJ}^{KL} K_{KL}^{MN} \times \left(\int_{\beta} F_\beta \phi_{\beta M}\phi_{\beta N} S'_\alpha(h_\alpha^{(1)})\right)$$

$$= -\bar{\rho}\left(\int_{\alpha} F_\alpha c_{\alpha x} c_{\alpha y}\phi_{\alpha A}\phi_{\alpha B}\right) K_{AB}^{IJ}\Theta_{IJ}^{KL} K_{KL}^{MN}\Theta_{MN}^{OP} K_{OP}^{QR} \times \left(\int_{\beta} F_\beta \phi_{\beta Q}\phi_{\beta R} S'_\beta(1)\right). \quad (75)$$

The series of the above form can be formally resummed in order to obtain a renormalized "Dyson" equation for the stress:

$$\sigma_{xy} = -\bar{\rho}\left(\int_{\alpha} F_\alpha c_{\alpha x} c_{\alpha y}\phi_{\alpha A}\phi_{\alpha B}\right) K_{AB}^{IJ}(\mathcal{I}_{IJ}^{MN} - \Theta_{IJ}^{KL} K_{KL}^{MN})^{-1} \times \left(\int_{\beta} F_\beta \phi_{\beta M}\phi_{\beta N} S'_\beta(1 + h_\alpha^{(1)} + h_\alpha^{(2)} + \cdots)\right), \quad (76)$$

where $\mathcal{I}_{IJ}^{MN}$ is the identity matrix. The transformation from a nondimensional to a dimensional stress is easily effected using the same procedure as that used for going from Eq. (64) to Eq. (65)

Note that the above relation can also be used for calculating the flux of mass due to a temperature gradient and vice versa. In a similar manner, the normal stresses σ_{xx}, σ_{yy}, and σ_{zz}, and the normal stress differences in a shear flow, can also be evaluated using expressions very similar to Eq. (76); the only modification required is that for σ_{xx}, for example, the first term on the right-hand is $-\rho\int_\alpha F_\alpha c_{\alpha x}^2 \phi_{\alpha A}\phi_{\alpha B}$ instead of $-\rho\int_\alpha F_\alpha c_{\alpha x} c_{\alpha y}\phi_{\alpha A}\phi_{\alpha B}$.

The other fluxes such as the mass and heat flux in the presence of concentration and temperature gradients can be easily evaluated using the analogs of Eq. (76). For this purpose, we need to define two quantities, the microscopic flux in terms of particle positions and velocities corresponding to the macroscopic flux and the perturbation to the Boltzmann equation due to the imposition of the macroscopic field gradient, equivalent to the operator S'_α in Eqs. (76) and (71). For example, if we consider a binary mixture, the equivalent of the microscopic flux $\mathbf{j}_i$ for component i in the mixture and the operator $S'_{\alpha i}$ are

$$\mathbf{j}_i = \bar{\rho}_i \mathbf{c}_{\alpha i},$$

$$S'_{\alpha i} = -(\nabla \bar{\rho}_i)\mathbf{c}_{\alpha i}, \quad (77)$$

where ρ_i is the mass density of component i. In a similar manner, for a temperature gradient, the microscopic flux $\mathbf{j}_T$ and the operator $S'_{\alpha T}$ are

$$\mathbf{j}_T = \frac{1}{2}\bar{\rho} c_\alpha^2 \mathbf{c}_\alpha,$$

$$S'_{\alpha T} = (\nabla T)\mathbf{c}_\alpha\left(\frac{c_\alpha^2}{2} - \frac{5}{2}\right). \quad (78)$$

With this general formulation, the macroscopic flux $\mathbf{J}$ corresponding to a macroscopic field gradient can be written as

$$\mathbf{J} = \left(\int_{\alpha} F_\alpha \mathbf{j}\phi_{\alpha A}\phi_{\alpha B}\right) K_{AB}^{IJ}(\mathcal{I}_{IJ}^{MN} - \Theta_{IJ}^{KL} K_{KL}^{MN})^{-1} \times \left(\int_{\beta} F_\beta \phi_{\beta M}\phi_{\beta N} S'_\beta(1 + h_\alpha^{(1)} + h_\alpha^{(2)} + \cdots)\right), \quad (79)$$

where Θ_{IJ}^{KL} is given by Eq. (71), with the operator S' given by Eqs. (77) and (78) for the mass and heat diffusion, respectively.

IV. GREEN-KUBO FORMULA AND DYNAMICAL TRANSITION

Expression (64) can be used to evaluate the shear viscosity. It is useful to first compare expression (64) with the result of the Green-Kubo formula for the shear viscosity:

$$\mu = \lim_{\mathbf{k}\to 0} \frac{1}{TV}\int_0^\infty dt \langle \sigma_{xy}(\mathbf{k},t)\sigma_{yx}(-\mathbf{k},0)\rangle, \tag{80}$$

where the $k\to 0$ limit is taken for the shear stress on the right-hand side. The microscopic shear stress for a system of N particles can be written as

$$\sigma_{xy}(\mathbf{k},t) = -\sum_{\alpha=1}^{N} c_{\alpha x}c_{\alpha y}\exp(\imath \mathbf{k}\cdot\mathbf{x}_\alpha). \tag{81}$$

In the limit $\mathbf{k}\to 0$ and for large N, the above expression for the stress can be written as

$$\lim_{\mathbf{k}\to 0}\sigma_{xy}(\mathbf{k},t) = -\sum_{\alpha=1}^{N} c_{\alpha x}c_{\alpha y}. \tag{82}$$

If the fluctuating velocity field is defined as

$$\tilde{u}_x(\mathbf{k}') = \sum_{\alpha=1}^{N} c_{\alpha x}\exp[-\imath \mathbf{k}'\cdot(\mathbf{x}-\mathbf{x}_\alpha)], \tag{83}$$

then the integral

$$\begin{aligned}\int_{\mathbf{k}'} \tilde{u}_x(\mathbf{k}')\tilde{u}_y(-\mathbf{k}') &= \sum_{\alpha=1}^{N}\sum_{\beta=1}^{N} c_{\alpha x}c_{\beta y}\int_{\mathbf{k}'} \exp[\imath \mathbf{k}'\cdot(\mathbf{x}_\alpha-\mathbf{x}_\beta)] \\ &= \sum_{\alpha=1}^{N}\sum_{\beta=1}^{N} c_{\alpha x}c_{\beta y}\delta(\mathbf{x}_\alpha-\mathbf{x}_\beta) \\ &= \sum_{\alpha=1}^{N} c_{\alpha x}c_{\alpha y},\end{aligned} \tag{84}$$

where the last step in the above equation follows from the fact that only one particle can occupy the position $\mathbf{x}_\alpha$. Inserting this into the Green-Kubo expression, we obtain

$$\mu = \frac{1}{\bar{T}V}\int_0^\infty dt\int_{\mathbf{k}'}\int_{\mathbf{k}''}\langle \tilde{u}_x(\mathbf{k}',t)\tilde{u}_y(-\mathbf{k}',t)\tilde{u}_x(\mathbf{k}'',0)\tilde{u}_y(-\mathbf{k}'',0)\rangle. \tag{85}$$

The velocities can be expressed in terms of the normal modes of the hydrodynamic matrix, using Eq. (54) as

$$\tilde{u}_x(\mathbf{k}',t)\tilde{u}_y(\mathbf{k}',t) = \frac{1}{2}\Gamma_{IJ}\bar{T}\phi_I(\mathbf{k}',t)\phi_J(-\mathbf{k}',t), \tag{86}$$

where ϕ_I is the matrix of macroscopic density, velocity, and temperature fields corresponding to the matrix $\phi_{\alpha I}$, which is a function of the particle velocities, and Γ_{IJ} is given in Eq. (54). This is inserted into the expression of the velocity to obtain

$$\mu = \frac{1}{2V}\Gamma_{IJ}\int_0^\infty dt\int_{\mathbf{k}'}\int_{\mathbf{k}''}\langle \phi_I(\mathbf{k}',t)\phi_J(-\mathbf{k}',t)u_x(\mathbf{k}'',0) \times u_y(-\mathbf{k}'',0)\rangle. \tag{87}$$

The integral in the above equation (87) could also be evaluated with $\mathbf{k}'$ defined as a wave vector at time t or at time $t=0$ in the rotating reference frame. This is because, from Eq. (4), it can easily be verified that the Jacobian for the transformation from the time-independent wave vector (k_x,k_y,k_z) to the time-dependent wave vector $(k_x,k_y-\dot{\gamma}tk_x,k_z)$ is equal to 1. We prefer to carry out the integral over the wave vector $\mathbf{k}'$ at $t=0$, and so the equation for the viscosity, Eq. (87), can also be written as

$$\mu = \frac{1}{2V}\Gamma_{IJ}\int_0^\infty dt\int_{\mathbf{k}'(0)}\int_{\mathbf{k}''(0)}\langle \phi_I(\mathbf{k}'(t),t) \times \phi_J(-\mathbf{k}'(t),t)u_x(\mathbf{k}''(0),0)u_y(-\mathbf{k}''(0),0)\rangle. \tag{88}$$

We express ϕ_I in terms of the hydrodynamic modes of the linearized Navier-Stokes operator:

$$\phi_I(\mathbf{k},t) = E_{IJ}(\mathbf{k},t)\xi_J(\mathbf{k},t). \tag{89}$$

When this is inserted into Eq. (87), we obtain

$$\mu = \frac{1}{2V}\Gamma_{IJ}\int_0^\infty dt\int_{\mathbf{k}'(0)}\int_{\mathbf{k}''(0)} E_{IP}(\mathbf{k}'(t),t)E_{JQ}(-\mathbf{k}'(t),t) \times\langle \xi_P(\mathbf{k}'(t),t)\xi_Q(-\mathbf{k}'(t),t)u_x(\mathbf{k}''(0),0)u_y(-\mathbf{k}''(0),0)\rangle. \tag{90}$$

The hydrodynamic modes of the linearized Navier-Stokes equations satisfy the orthogonality relation

$$\xi_P(\mathbf{k}'(t),t) = \xi_P(\mathbf{k}'(0),0)\exp\left(-\int_0^t dt'\lambda_P(\mathbf{k}'(t'))\right). \tag{91}$$

The final expression for the shear viscosity is

$$\begin{aligned}\mu &= \frac{1}{2T}\Gamma_{IJ}\int_{\mathbf{k}'(0)}\int_{\mathbf{k}''(0)}\int_0^\infty dt\, E_{IP}(\mathbf{k}'(t),t)E_{JQ}(-\mathbf{k}'(t),t) \\ &\times\exp\left[-\left(\int_0^t dt'[\lambda_P(\mathbf{k}'(t'))+\lambda_Q(-\mathbf{k}'(t'))]\right)\right] \\ &\times\langle \xi_P(\mathbf{k}'(0),0)\xi_Q(-\mathbf{k}'(0),0)u_x(\mathbf{k}'',0)u_y(-\mathbf{k}'',0)\rangle \\ &= \frac{1}{2TV}\Gamma_{IJ}\int_{\mathbf{k}'(0)}\int_{\mathbf{k}''(0)}\langle \phi_P(\mathbf{k}',0)\phi_Q(-\mathbf{k}',0)u_x(\mathbf{k}'',0) \\ &\times u_y(-\mathbf{k}'',0)\rangle E^{-1}_{PR}(\mathbf{k}''(0),0)E^{-1}_{QS}(-\mathbf{k}''(0),0) \\ &\times\int_0^\infty dt\, E_{IP}(\mathbf{k}'(t),t)E_{JQ}(-\mathbf{k}'(t),t) \\ &\times\exp\left(-\int_0^t dt'[\lambda_P(\mathbf{k}'(t'))+\lambda_Q(-\mathbf{k}'(t'))]\right).\end{aligned} \tag{92}$$

It can easily be shown that, to leading order in small ϵ,

$$\langle \phi_P(\mathbf{k}',0)\phi_Q(-\mathbf{k}',0)u_x(\mathbf{k}'',0)u_y(-\mathbf{k}'',0)\rangle = \Gamma_{PQ}\delta(\mathbf{k}'+\mathbf{k}'')\delta(-\mathbf{k}'-\mathbf{k}'')\bar{T} = \Gamma_{PQ}\delta(\mathbf{k}'+\mathbf{k}'')\bar{T}V, \tag{93}$$

where T is the dynamical temperature. Note that there are corrections to Eq. (93) due to the shear-induced anisotropy in the distribution function, and therefore Eq. (93) is valid only to leading order in the limit of small ϵ. Using Eq. (93), the Green-Kubo relation for the shear viscosity reduces to

$$\mu = \frac{\bar{T}}{2}\Gamma_{IJ}\Gamma_{KL}\int_{\mathbf{k}'(0)} E^{-1}_{PK}(-\mathbf{k}'(0),0)E^{-1}_{QL}(\mathbf{k}'(0),0) \times \int_0^\infty dt\, E_{IP}(\mathbf{k}'(t),t)E_{JQ}(-\mathbf{k}'(t),t) \times \exp\left(-\int_0^t dt'[\lambda_P(\mathbf{k}'(t')) + \lambda_Q(-\mathbf{k}'(t'))]\right). \tag{94}$$

By effecting the transformation $t \to -t$ and $t' \to -t'$, we obtain the final expression for the viscosity:

$$\mu = \frac{T}{2}\Gamma_{IJ}\Gamma_{KL}\int_{\mathbf{k}'(0)} E^{-1}_{PK}(-\mathbf{k}',0)E^{-1}_{QL}(\mathbf{k}',0) \times \int_{-\infty}^0 dt\, E^{\dagger}_{PI}(\mathbf{k}'(-t),-t)E^{\dagger}_{QJ}(-\mathbf{k}'(-t),-t) \times \exp\left(-\int_t^0 dt'[\lambda_P(\mathbf{k}'(-t')) + \lambda_Q(-\mathbf{k}'(-t'))]\right). \tag{95}$$

There are a few notable features of the comparison of Eq. (95) from the Green-Kubo relation and Eq. (64) from the ring-kinetic equation.

(i) Clearly, the expression (95) is identical to Eq. (64) derived from the ring-kinetic expansion if the eigenvalues λ_P and eigenvectors E_{PQ} are independent of time. Since they are time dependent in the present analysis, the correlations in Eq. (95) have to be taken with time reversed. Note that reversing time in the present system is equivalent to reversing the direction of the mean shear—that is, effecting the transformation $\dot{\gamma} \to -\dot{\gamma}$. Therefore, the direction of shear has to be chosen carefully while interpreting the Green-Kubo relations. If Eq. (80) is to be used, then the correlation functions have to be calculated with shear in the opposite direction to the one used for the response function. In the present result for the shear viscosity, Eq. (95), this distinction does not make a difference to the end result, since we are integrating over all wave vectors anyway. However, it may make a difference in the event we are calculating a wave-vector-dependent viscosity.

(ii) The comparison of Eqs. (95) and (64) shows that the equivalent of the temperature in the Green-Kubo relation is the average translational temperature in the sheared state in ring-kinetic equation. Note that the temperature in the sheared state is not a thermodynamic temperature, but is determined by a balance between the rates of production of energy due to mean shear and dissipation due to inelastic collisions. In contrast, the thermodynamic temperature of the system is to be used in the Green-Kubo expression.

(iii) The other important difference to note is that the Green-Kubo relation equation (95), as defined, is calculated using correlations in the equilibrium (unsheared) state. The result from the ring-kinetic equation (64) is for the sheared state. This makes a significant difference in the results obtained by the two calculations, since the hydrodynamic modes for the sheared state in the long-wave limit are significantly different from those for the equilibrium state. Therefore, an important result of the present analysis is that to leading order in small ϵ, the Green-Kubo relation is valid for the shear viscosity provided the decay of the hydrodynamic modes in the sheared state is used in the calculation. This is, of course, conditional on there being no further corrections due to three-particle and higher correlations. This is discussed using diagrammatic expansions in the next section.

(iv) Finally, the Green-Kubo relation is only the leading-order approximation in the limit of small ϵ. There are higher-order corrections, which can be systematically calculated using the ϵ expansion, and the present analysis provides the framework for calculating these.

The series solution has been extended to higher orders. In addition, we have been able to explicitly resum the series to obtain Eq. (76). The term $(\mathcal{I}^{MN}_{IJ} - \Theta^{KL}_{IJ}K^{MN}_{KL})^{-1}$ on the right-hand side of Eq. (76) is of interest. If the matrix $(\mathcal{I}^{MN}_{IJ} - \Theta^{KL}_{IJ}K^{MN}_{KL})^{-1}$ has a zero determinant, then the stress diverges even though the strain rate is finite and all the terms in the Chapman-Enskog expansion for the single-particle distributing function are finite. It should be noted that the matrix $(\mathcal{I}^{MN}_{IJ} - \Theta^{KL}_{IJ}K^{MN}_{KL})^{-1}$ depends only on the hydrodynamic modes of the linearized Boltzmann equation for a shear flow and is independent of the strength of the corrections $h^{(1)}_\alpha, h^{(2)}_\alpha, \ldots$ for the Chapman-Enskog solution for the velocity distribution function. In addition, the matrices Θ^{KL}_{IJ} and K^{MN}_{KL} are functions of the type of forcing through the operator S'_α in Eqs. (61) and (71). A steady shear flow has been used as the forcing in the present case, but we have indicated how the analysis could be extended to other types of forcing. It is clear that if the matrix $(\mathcal{I}^{MN}_{IJ} - \Theta^{KL}_{IJ}K^{MN}_{KL})$ has a zero determinant, the divergence of the stress response is specific to the type of forcing used. Therefore, this represents a jamming or slowing down of the sheared steady state, which is a nonequilibrium transition, even though there is no nearby equilibrium transition.

V. VISCOSITY FOR AN INELASTIC FLUID

We return to the expression for the viscosity, Eq. (64). The result of the expression depends on the range of wave vectors we consider while calculating the wave vector integral. If we consider the range $k \gg \epsilon$, where energy is treated as a conserved variable, there are five hydrodynamic modes. Equation (64) states that the ring-kinetic contributions to the viscosity are due to coupling between different hydrodynamic modes. There are two types of coupling possible, those in which $(\lambda_P + \lambda_Q) \sim k$ and those in which $(\lambda_P + \lambda_Q) \sim k^2$. The latter result in the most singular contributions to

the viscosity, and so we shall examine these in some detail. These result in integrals for the scaled viscosity of the type

$$\mu \sim \int_{-\infty}^{0} dt \int_{\mathbf{k}} A \exp\left(- \int_{t}^{0} \lambda'(k^2 - 2\dot{\gamma} k_x k_y t + \dot{\gamma}^2 k_x^2 t^2) \right)$$
$$\sim \int_{-\infty}^{0} dt (2\pi)^{-D} \int k^{D-1} dk \int d\Omega\, A$$
$$\times \exp[\lambda' k^2 (t - \dot{\gamma} \hat{k}_x \hat{k}_y t^2 + \dot{\gamma}^2 \hat{k}_x^2 t^3/3)], \tag{96}$$

where A is a constant (correct to leading order in small k), $\lambda' = (\lambda_P + \lambda_Q)/k^2$ is a constant, and the time-dependent terms in the exponential arise due to the turning of the wave vector with time. In Eq. (96), the wave vector integral has been separated into two parts, one of which is the magnitude of the wave vector k and the other is the integral over the solid angle $d\Omega$ of the orientation of the wave vector with respect to a fixed axis. Note that the integral over the magnitude of the wave vector is proportional to k^D, where D is the dimensionality of the system.

The integral over k can be explicitly carried out, to obtain

$$\mu \sim \int_{t}^{0} dt (2\pi)^{-1} \int d\Omega \frac{A}{[\lambda' k^2 (t - \dot{\gamma} \hat{k}_x \hat{k}_y t^2 + \dot{\gamma}^2 \hat{k}_x^2 t^3/3)]^{D/2}}. \tag{97}$$

Next, we evaluate the time integral in the above expression. If the strain rate were zero, then the time integral would have a logarithmic divergence in two dimensions and would have a $t^{-1/2}$ decay in three dimensions. However, when there is a nonzero strain rate, this divergence gets cut off at $t \sim \dot{\gamma}^{-1}$ due to the decrease proportional to t^3 in the long-time limit and there is a much faster decay for $t \gg \dot{\gamma}^{-1}$, which makes a negligible contribution to the integral. Due to this, the viscosity is found to be proportional to $\ln(\dot{\gamma})$ in two dimensions and proportional to $\dot{\gamma}^{1/2}$ in three dimensions. Quantitative results can be obtained by calculating the above integrals exactly, as was done in Ernst *et al.* [11] using a different procedure, which was to solve differential equations in wave number space at steady state. We have carried out numerical results using the procedure outlined here and using the eigenvalues (32) and eigenfunctions (A7) in Part I and verified that the same numerical results are obtained.

The above results are valid only for $k \gg \epsilon$. For the domain $k \ll \epsilon$, it is necessary to use the eigenvalues (44) and eigenvectors (A10) in Part I. If we restrict our attention to the leading approximation $\mathbf{E}^{(0)}$ for the eigenvalues, then we obtain, quite easily, the following result for the viscosity:

$$\mu = \int_{\mathbf{k}} \int_{-\infty}^{0} dt \frac{2 s_0(t)}{3 s_0(0)} \{\exp(S_0 + S_1 - 2S_2) + (-1)^{2/3}$$
$$\times \exp[\,(-1)^{4/3} S_0) + (-1)^{2/3} S_1 - 2S_2] + (-1)^{4/3}$$
$$\times \exp[(-1)^{2/3} S_0 + (-1)^{4/3} S_1 - 2S_2]\}, \tag{98}$$

where S_0, S_1, and S_2 are given in Eq. (44) of Part I. We have also calculated the contributions to the normal stress differences and the total temperature due to correlations using the eigenvalues (44) and eigenfunctions (A10) of Part I and found these to be zero.

In the limit $k_x \to 0$, the growth rates and the eigenvectors of the hydrodynamic modes are given by Eqs. (32) and (A7) of Part I. In this case, the result equivalent to Eq. (98) for the viscosity is

$$\mu = \int_{\mathbf{k}} \int_{0}^{\infty} dt \left[\exp\left(\int_{0}^{t} dt' \lambda_1(t') \right) + \exp\left(\int_{0}^{t} dt' \lambda_2(t') \right) \right]^2, \tag{99}$$

where λ_1 and λ_2 are given in Eq. (32) of Part I.

Equations (98) and (99) contain the velocity autocorrelation functions in the x and y directions in Eqs. (64) of Part I. Therefore, the scaling behavior of these autocorrelation functions obtained in Eqs. (58) and (59) of Part I can be used to estimate the integrals. The first point to note is that the leading order decay rate in the limit $k \to 0$ is proportional to $k^{2/3}$ in Eq. (44) of Part I. This is in contrast to the decay rate proportional to k^2 for an elastic system and for the case $k \gg \epsilon$ in Eq. (96). Even in three dimensions, there is no mode-coupling contribution from the transverse mode with eigenvalue $\lambda_4 \propto k^2$ in Eq. (44) of Part I. Therefore, the divergence in the viscosity in two dimensions and the divergence in the Burnett coefficients in three dimensions are not present for an inelastic fluid. Equation (58) of Part I indicates that the decay of fluctuations proportional to $k^{2/3}$ translates to a decay in the velocity autocorrelation functions proportional to $t^{-5D/4}$ in a sheared inelastic fluid in the long-time limit. Even in the gradient vorticity plane with $k_x \to 0$, Eq. (59) of Part I shows that the autocorrelation function decays proportional to $t^{-(D+1)}$. Therefore, the difficulties with divergences of the transport coefficients are not present in an inelastic fluid. It is important to note that the difference is due to the difference in the nature of the hydrodynamic modes, and not due to a significant modification in the expression for the viscosity; as we have shown earlier, the Green-Kubo expression does accurately predict the viscosity correct to leading order in an expansion in the parameter ϵ, provided the correlation functions are calculated in the uniformly sheared state.

The second implication of Eq. (98) is that there could be a significant modification in the viscosity, because of the unstable nature of the hydrodynamic modes. It is clear that if only the contribution S_0 is retained in the exponentials in Eq. (98), then there is a divergence in either one or two of the exponential terms in the limit $t \to -\infty$, depending on the sign of s_0. However, in the long-time limit, the term S_2 increases proportional to t^3 due to the dependence of wave vector on time, and this renders the integrals convergent for $t \gg \dot{\gamma}^{-1}$. However, the transient exponential increase would result in a significant numerical contribution to the system viscosity.

VI. HIGHER-ORDER CORRELATIONS

The analysis of the previous section can be extended diagrammatically to equations for higher-order correlation functions. Here, the particle coordinates are denoted by the Greek subscripts α, β, etc. The notation F_{α}^{CE} is used to represent the Chapman-Enskog solution for the single-particle distribution

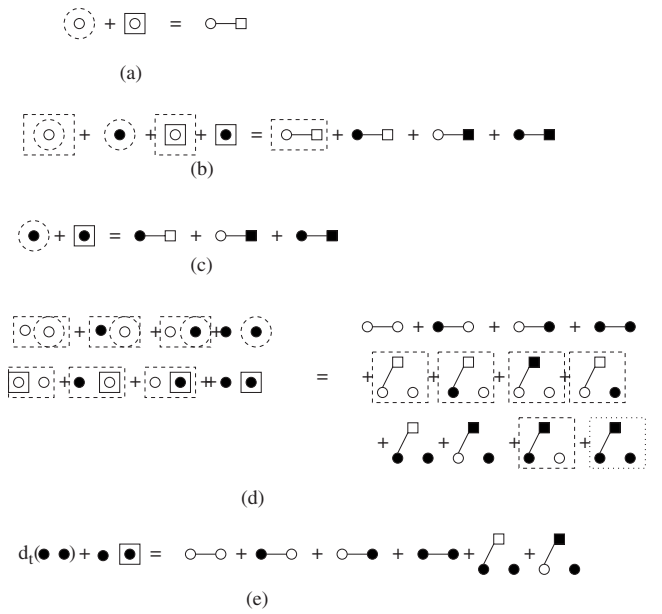

FIG. 1. Diagrams for the one- and two-particle correlation functions.

function, which is the solution for the Boltzmann equation. The effect of correlations is included by expanding the single-, two-, and many-particle distributions as

$$f_\alpha = \bar{\rho} F_\alpha^{CE}(1 + g_\alpha),$$

$$f_{\alpha\beta} = \bar{\rho}^2 F_\alpha^{CE} F_\beta^{CE}(1 + g_\alpha + g_\beta + g_{\alpha\beta}),$$

$$f_{\alpha\beta\gamma} = \bar{\rho}^3 F_\alpha^{CE} F_\beta^{CE} F_\gamma^{CE}(1 + g_\alpha + g_\beta + g_\gamma + g_{\alpha\beta} + g_{\beta\gamma} + g_{\alpha\gamma} + g_{\alpha\beta\gamma}). \tag{100}$$

The diagrammatic scheme that we use is as follows. First, we consider the diagrammatic representation of the Boltzmann equation:

$$(\partial_t + S_\alpha) F_\alpha^{CE} = \int_\beta C_{\alpha\beta}[F_\alpha^{CE} F_\beta^{CE}]. \tag{101}$$

The diagrammatic representation of the above equation is shown in Fig. 1(a). The open circle represents a "root node" (distribution function F_α^{CE}) which is not integrated over the velocities of particle α. The square circumscribing F_α^{CE} on the left is the operator S_α in Eq. (14), while the time derivative is shown by the superscribed circles. It should be noted that $\partial_t(\circ)$ represents $\partial_t(F_\alpha^{CE})$, which is equal to zero because the single-particle distribution function is identically zero. However, we retain this in the diagram in order to cancel out the terms in the higher-order correlation function. On the right-hand side, the open square represents a "field node" (distribution function F_β^{CE} which is integrated over), and the line joining the root and field nodes is the collision operator. So the symbol on the right-hand side of Fig. 1(a) represents $C_{\alpha\beta}[F_\alpha^{CE} F_\beta^{CE}]$.

The equation for the single-particle distribution function, which includes the effect of correlations, is

$$(\partial_t + S_\alpha)[F_\alpha^{CE}(1 + g_\alpha)] = \int_\beta C_{\alpha\beta}[F_\alpha^{CE} F_\beta^{CE}(1 + g_\alpha + g_\beta + g_{\alpha\beta})]. \tag{102}$$

This equation for the single-particle distribution function is shown in Fig. 1(b). In this equation, on both the left- and right-hand sides, a solid circle on the left-hand side represents $F_\alpha^{CE} g_\alpha$. It should be noted that the symbol $\partial_t(\bullet)$ is $\partial_t(F_\alpha^{CE} g_\alpha)$, which is zero because g_α is independent of time. On the right-hand side, a solid circle represents and an open square connected by a line represents $\int_\beta C_{\alpha\beta}[F_\alpha^{CE} F_\beta^{CE} g_\alpha]$, and a solid square and an open circle connected by a line represent $\int_\beta C_{\alpha\beta}[F_\alpha^{CE} F_\beta^{CE} g_\beta]$, while a solid circle and a solid square connected by a line represent $\int_\beta C_{\alpha\beta}[F_\alpha^{CE} F_\beta^{CE} g_{\alpha\beta}]$. Since all terms in the expansion are linear in the correlation functions g, the number of solid symbols indicates the number of particles connected by the correlation without loss of ambiguity. In Fig. 1(b), the terms enclosed in the dotted rectangles are identical to those appearing in Fig. 1(a), and so these can be removed by subtracting the terms in Fig. 1(a) from the terms in Fig. 1(b). With this, we are left with the reduced equation, Fig. 1(c), which provides the reduced equation for the single-particle correlation function:

$$(\partial_t + S_\alpha)(F_\alpha^{CE} g_\alpha) = \int_\beta C_{\alpha\beta}[F_\alpha^{CE} F_\beta^{CE}(g_\alpha + g_\beta + g_{\alpha\beta})]. \tag{103}$$

The complete equation for the two-particle distribution function,

$$\begin{aligned}(\partial_t + S_\alpha + S_\beta)[F_\alpha^{CE} F_\beta^{CE}(1 + g_\alpha + g_\beta + g_{\alpha\beta})] &= \bar{\rho}^{-1}\delta(\mathbf{x}_{\alpha\beta}) C_{\alpha\beta}[F_\alpha^{CE} F_\beta^{CE}(1 + g_\alpha + g_\beta + g_{\alpha\beta})] \\ &+ \int_\gamma C_{\alpha\gamma}[F_\alpha^{CE} F_\beta^{CE} F_\gamma^{CE}(1 + g_\alpha + g_\beta + g_\gamma + g_{\alpha\beta} + g_{\alpha\gamma} + g_{\beta\gamma} + g_{\alpha\beta\gamma})] + \int_\gamma C_{\beta\gamma}[F_\alpha^{CE} F_\beta^{CE} F_\gamma^{CE}(1 + g_\alpha + g_\beta + g_\gamma + g_{\alpha\beta} \\ &+ g_{\alpha\gamma} + g_{\beta\gamma} + g_{\alpha\beta\gamma})],\end{aligned} \tag{104}$$

is shown in Fig. 1(d). The diagrams on the left-hand side are related to the terms in the above equation as follows.

(i) The first term on the left represents the time derivative $F_\alpha^{CE}\partial_t(F_\beta^{CE}) + F_\beta^{CE}\partial_t(F_\alpha^{CE})$.

(ii) The second represents the time derivative $F_\alpha^{CE} g_\alpha \partial_t(F_\beta^{CE}) + F_\beta^{CE} g_\beta \partial_t(F_\alpha^{CE})$. Note that this is an unlabeled diagram, which includes both terms.

(iii) The third term is the time derivative $F_\alpha^{CE}\partial_t(F_\beta^{CE} g_\beta) + F_\beta^{CE}\partial_t(F_\alpha^{CE} g_\alpha)$.

(iv) The fourth is the time derivative $\partial_t(F_\alpha^{CE} F_\beta^{CE} g_{\alpha\beta})$.

(v) The squares represent the S operator acting on the terms on the left-hand side of the three-particle equation. The fifth diagram on the left-hand side, which contains two open circles with a square around one of the open circles, represents $S_\alpha(F_\alpha^{CE} F_\beta^{CE}) + S_\beta(F_\alpha^{CE} F_\beta^{CE})$. Since the operator S_α acting

on F_β^{CE} gives zero (the coordinates of particles α and β are independent), this term can be factored as $F_\beta^{CE}S_\alpha(F_\alpha^{CE}) + F_\alpha^{CE}S_\beta(F_\beta^{CE})$.

(vi) The sixth diagram on the left represents $S_\alpha(F_\alpha^{CE}F_\beta^{CE}g_\beta) + S_\beta(F_\alpha^{CE}F_\beta^{CE}g_\alpha)$. This can also be written as $F_\beta^{CE}g_\beta S_\alpha(F_\alpha^{CE}) + F_\alpha^{CE}g_\alpha S_\beta(F_\beta^{CE})$, since S_α acting on $F_\beta^{CE}g_\beta$ gives zero.

(vii) The seventh diagram on the left is $S_\alpha(F_\alpha^{CE}F_\beta^{CE}g_\alpha) + S_\beta(F_\alpha^{CE}F_\beta^{CE}g_\beta)$. This can also be written as $F_\beta^{CE}S_\alpha(F_\alpha^{CE}g_\alpha) + F_\alpha^{CE}S_\beta(F_\beta^{CE}g_\beta)$, since S_α acting on F_β^{CE} is equal to zero.

(viii) The eighth diagram on the left is $S_\alpha(F_\alpha^{CE}F_\beta^{CE}g_{\alpha\beta}) + S_\beta(F_\alpha^{CE}F_\beta^{CE}g_{\alpha\beta})$.

On the right-hand side, there are collision terms involving the two-particle distribution functions $C_{\alpha\beta}[F_\alpha^{CE}F_\beta^{CE}]$, $C_{\alpha\beta}[F_\alpha^{CE}F_\beta^{CE}g_\alpha]$, and $C_{\alpha\beta}[F_\alpha^{CE}F_\beta^{CE}g_{\alpha\beta}]$. In addition, there are three-particle terms with binary collisions, which involve the single-particle, two-particle, and the three-particle correlation functions. We consider the three-particle terms first, since many of these terms are removed when a reduced equation for the two-particle distribution is derived.

(i) The 5th term on the right-hand side represents $\int_\gamma C_{\alpha\gamma}[F_\alpha^{CE}F_\beta^{CE}F_\gamma^{CE}] + \int_\gamma C_{\beta\gamma}[F_\alpha^{CE}F_\beta^{CE}F_\gamma^{CE}]$. Since the collision operator $C_{\alpha\gamma}$ does not alter the distribution function F_β^{CE}, this term can also be written as $F_\beta^{CE}\int_\gamma C_{\alpha\gamma}[F_\alpha^{CE}F_\gamma^{CE}] + F_\alpha^{CE}\int_\gamma C_{\beta\gamma}[F_\gamma^{CE}F_\gamma^{CE}]$.

(ii) The 6th diagram on the right-hand side represents $\int_\gamma C_{\alpha\gamma}[F_\alpha^{CE}F_\beta^{CE}F_\gamma^{CE}g_\alpha] + \int_\gamma C_{\beta\gamma}[F_\alpha^{CE}F_\beta^{CE}F_\gamma^{CE}g_\beta]$, which can also be written as $F_\beta^{CE}\int_\gamma C_{\alpha\gamma}[F_\alpha^{CE}F_\gamma^{CE}g_\alpha] + F_\alpha^{CE}\int_\gamma C_{\beta\gamma}[F_\beta^{CE}F_\gamma^{CE}g_\beta]$.

(iii) The 7th diagram on the right-hand side represents $\int_\gamma C_{\alpha\gamma}[F_\alpha^{CE}F_\beta^{CE}F_\gamma^{CE}g_\gamma] + \int_\gamma C_{\beta\gamma}[F_\alpha^{CE}F_\beta^{CE}F_\gamma^{CE}g_\gamma]$, which can be factored into $F_\beta^{CE}\int_\gamma C_{\alpha\gamma}[F_\alpha^{CE}F_\gamma^{CE}g_\gamma] + F_\alpha^{CE}\int_\gamma C_{\beta\gamma}[F_\beta^{CE}F_\beta^{CE}g_\gamma]$.

(iv) The 8th term on the right diagram on the right-hand side represents $\int_\gamma C_{\alpha\gamma}[F_\alpha^{CE}F_\beta^{CE}F_\gamma^{CE}g_\beta] + \int_\gamma C_{\beta\gamma}[F_\alpha^{CE}F_\beta^{CE}F_\gamma^{CE}g_\alpha]$, which can be factored into $F_\beta^{CE}g_\beta\int_\gamma C_{\alpha\gamma}[F_\alpha^{CE}F_\gamma^{CE}] + F_\alpha^{CE}g_\alpha\int_\gamma C_{\beta\gamma}[F_\beta^{CE}F_\gamma^{CE}]$.

(v) The 9th diagram on the right-hand side is $\int_\gamma C_{\alpha\gamma}[F_\alpha^{CE}F_\beta^{CE}F_\gamma^{CE}g_{\alpha\beta}] + \int_\gamma C_{\beta\gamma}[F_\alpha^{CE}F_\beta^{CE}F_\gamma^{CE}g_{\alpha\beta}]$. These terms can be simplified further, but it turns out that these terms are retained in the reduced equation, and so we do not simplify these further.

(vi) The 10th term on the right-hand side is $\int_\gamma C_{\alpha\gamma}[F_\alpha^{CE}F_\beta^{CE}F_\gamma^{CE}g_{\alpha\gamma}] + \int_\gamma C_{\beta\gamma}[F_\alpha^{CE}F_\beta^{CE}F_\gamma^{CE}g_{\alpha\gamma}]$. This term also appears in the final reduced equation, and so we do not simplify this further.

(vii) The 11th term on the right-hand side is $\int_\gamma C_{\alpha\gamma}[F_\alpha^{CE}F_\beta^{CE}F_\gamma^{CE}g_{\alpha\gamma}] + \int_\gamma C_{\beta\gamma}[F_\alpha^{CE}F_\beta^{CE}F_\gamma^{CE}g_{\beta\gamma}]$. This term can be simplified to, $F_\beta^{CE}\int_\gamma C_{\alpha\gamma}[F_\alpha^{CE}F_\gamma^{CE}g_{\alpha\gamma}] + F_\alpha^{CE}\int_\gamma C_{\beta\gamma}[F_\beta^{CE}F_\gamma^{CE}g_{\beta\gamma}]$.

(viii) The final term on the right-hand side involved the three-particle correlation function, and this term is of the form $\int_\gamma C_{\alpha\gamma}[F_\alpha^{CE}F_\beta^{CE}F_\gamma^{CE}g_{\alpha\beta\gamma}] + \int_\gamma C_{\beta\gamma}[F_\alpha^{CE}F_\beta^{CE}F_\gamma^{CE}g_{\alpha\beta\gamma}]$. In the ring-kinetic approximation, this is neglected because it involves the three-particle correlation function.

Now we discuss the cancellation of the terms in Fig. 1(d). First, we add the first and fifth diagrams on the left-hand side and then subtract the 5th diagram on the right-hand side to obtain

$$\partial_t(F_\alpha^{CE}F_\beta^{CE}) + (S_\alpha + S_\beta)(F_\alpha^{CE}F_\beta^{CE}) - \int_\gamma C_{\alpha\gamma}[F_\alpha^{CE}F_\beta^{CE}F_\gamma^{CE}] + \int_\gamma C_{\beta\gamma}[F_\alpha^{CE}F_\beta^{CE}F_\gamma^{CE}]. \tag{105}$$

This can be rearranged to provide

$$F_\beta^{CE}\left(\partial_t F_\alpha^{CE} + S_\alpha(F_\alpha^{CE}) - \int_\gamma C_{\alpha\gamma}[F_\alpha^{CE}F_\gamma^{CE}]\right) + F_\alpha^{CE}\left(\partial_t F_\beta^{CE} + S_\beta(F_\beta^{CE}) - \int_\gamma C_{\beta\gamma}[F_\beta^{CE}F_\gamma^{CE}]\right). \tag{106}$$

It is easily seen that both the 1st and 2nd terms in the expression above are zero by comparing with the Boltzmann equation (101).

Next, we add the 2nd and 6th diagrams on the left-hand side and then subtract the 8th diagram on the right-hand side to obtain

$$F_\beta^{CE}g_\beta\partial_t(F_\alpha^{CE}) + F_\alpha^{CE}g_\alpha\partial_t(F_\beta^{CE}) + F_\beta^{CE}g_\beta S_\alpha(F_\alpha^{CE}) + F_\alpha^{CE}g_\alpha S_\beta(F_\beta^{CE}) - F_\beta^{CE}g_\beta\int_\gamma C_{\alpha\gamma}[F_\alpha^{CE}F_\gamma^{CE}] - F_\alpha^{CE}g_\alpha\int_\gamma C_{\beta\gamma}[F_\beta^{CE}F_\gamma^{CE}]. \tag{107}$$

This expression can be simplified to obtain

$$F_\beta^{CE}g_\beta\left(\partial_t(F_\alpha^{CE}) + S_\alpha(F_\alpha^{CE}) - \int_\gamma C_{\alpha\gamma}[F_\alpha^{CE}F_\gamma^{CE}]\right) + F_\alpha^{CE}g_\alpha\left(\partial_t(F_\beta^{CE}) + S_\beta(F_\beta^{CE}) - \int_\gamma C_{\beta\gamma}[F_\beta^{CE}F_\gamma^{CE}]\right). \tag{108}$$

It is clear, by comparing with the Boltzmann equation (101) that the above expressions are zero. Next, consider the expression obtained by adding the 3rd and 7th terms on the left-hand side, and subtracting the 6th, 7th, and 11th terms on the right-hand side. The expression obtained is

$$F_\beta^{CE}\partial_t(F_\alpha^{CE}g_\alpha) + F_\alpha^{CE}\partial_t(F_\beta^{CE}g_\beta) + F_\beta^{CE}S_\alpha(F_\alpha^{CE}g_\alpha) + F_\alpha^{CE}S_\beta(F_\beta^{CE}g_\beta) - F_\beta^{CE}\int_\gamma C_{\alpha\gamma}[F_\alpha^{CE}F_\gamma^{CE}g_\alpha] - F_\alpha^{CE}\int_\gamma C_{\beta\gamma}[F_\beta^{CE}F_\gamma^{CE}g_\beta] - F_\beta^{CE}\int_\gamma C_{\alpha\gamma}[F_\alpha^{CE}F_\gamma^{CE}g_\gamma] - F_\alpha^{CE}\int_\gamma C_{\beta\gamma}[F_\beta^{CE}F_\gamma^{CE}g_\gamma] - C_{\alpha\gamma}[F_\alpha^{CE}F_\beta^{CE}F_\gamma^{CE}g_{\alpha\gamma}] - C_{\beta\gamma}[F_\alpha^{CE}F_\beta^{CE}F_\gamma^{CE}g_{\beta\gamma}]. \tag{109}$$

It is easy to simplify the above expression to obtain

$$F_\beta^{CE}\Bigg(\partial_t(F_\alpha^{CE}g_\alpha)+S_\alpha(F_\alpha^{CE}g_\alpha) - \int_\gamma C_{\alpha\gamma}[F_\alpha^{CE}F_\gamma^{CE}(g_\alpha+g_\gamma+g_{\alpha\gamma})]\Bigg) + F_\alpha^{CE}\Bigg(\partial_t(F_\beta^{CE}g_\beta)+S_\beta(F_\beta^{CE}g_\beta) - \int_\gamma C_{\beta\gamma}[F_\beta^{CE}F_\gamma^{CE}(g_\beta+g_\gamma+g_{\beta\gamma})]\Bigg). \quad (110)$$

It is easy to verify that the above expression is zero due to the reduced equation for the single-particle correlation function (103).

From the above simplification, it is evident that the diagrams with the superscribed dashed rectangles are those that can be factored into the single-particle distribution functions in Fig. 1(d). As mentioned above, the term with the superscribed dotted rectangle contains the three-particle distribution function $g_{\alpha\beta\gamma}$. If this is neglected in the ring-kinetic approximation, we obtain the reduced equation for the three-particle distribution function, Fig. 1(e).

Using the diagrammatic method, we can write down some rules for obtaining the reduced nth-order correlation function.

(i) All time derivatives with one or more open circles on the left-hand side can be neglected, since they can be factored into terms that already appear in the lower-order distribution functions.

(ii) All streaming terms with one or more open circles on the left-hand side can be removed since they can be factored into terms appearing in the equations for lower-order correlation functions.

(iii) On the right-hand side, all collision terms involving $n+1$ nodes which contain a collision connection between an open circle and open square can be removed, since the collision operator is acting on g functions of particles not involved in the collision.

(iv) All collision terms involving $n+1$ particles which contain an open circle not involved in a collision can be removed, since it can be factored into two terms, one of which is the F^{CE} of the open circle and the second of which appears in a lower-order diagram.

(v) In the terms with $n+1$ nodes and n solid circles, there are two diagrams which survive when the above rules are applies; these are the diagrams in which the open circle is on one of the nodes linked by a collision.

(vi) In the terms with $n+1$ nodes and $n-1$ solid circles, there are no terms that survive; if the two open circles are linked by a collision, they are removed due to rule (iii), while if one of the open circles is not linked by a collision, it is removed due to rule (iv).

(vii) In the closure approximation, the collision term involving n solid circles and one solid square is removed, since this represents the $(n+1)$th-order correlation function.

A further simplification is obtained by combining the streaming operators on the left with the second and third collision operators on the right. If we use the definition of the operator L_α,

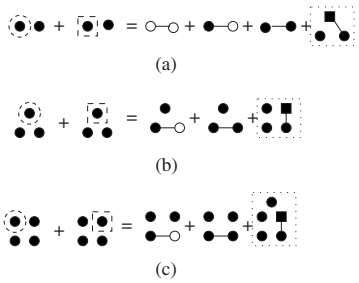

FIG. 2. Diagrams for the two-, three-, and four-particle correlation functions, (a), (b), and (c), respectively, in terms of the operator L_α.

$$L_\alpha(\psi)=S_\alpha(\psi)-\int_\gamma C_{\alpha\gamma}[(1+P_{\alpha\gamma})(F_\gamma\psi)], \quad (111)$$

where $P_{\alpha\gamma}$ is the "permutation operator" which changes the index α to γ. With this the equation for the single-particle correlation function is considerably simplified,

$$\left(\frac{\partial}{\partial t}+L_\alpha\right)(F_\alpha^{CE}g_\alpha)=\int_\gamma C_{\alpha\gamma}[F_\alpha^{CE}F_\gamma^{CE}g_{\alpha\gamma}], \quad (112)$$

and the equation for the two-particle correlation function is given by

$$\left(\frac{\partial}{\partial t}+L_\alpha+L_\beta\right)(F_\alpha^{CE}F_\beta^{CE}g_{\alpha\beta}) = \bar{\rho}^{-1}\delta(\mathbf{x}_{\alpha\beta})C_{\alpha\beta}[F_\alpha^{CE}F_\beta^{CE}(1+g_\alpha+g_\beta+g_{\alpha\beta})]. \quad (113)$$

Diagrammatically, this simplification is shown in Fig. 2, where the dashed superscribed rectangle on the left represents the operator L_α. The transformation from S_α to L_α is equivalent to removing two diagrams with $n+1$ nodes and $n-1$ solid circles not connected by collision links and either one open circle and one solid square connected by a collision link *or* one open square and solid circle connected by a collision link. Therefore, the only diagram on the right-hand side with $n+1$ nodes is the one with $n+1$ solid circles. Equation (20), which is shown in diagram 2(a) with the last term on the right neglected, is the repeated ring equation. If we neglect the terms proportional to g_α, g_β, and $g_{\alpha\beta}$ in the integral on the right-hand side of the above equation, we obtain the simple ring-kinetic equation, which is a closed and linear equation for the two-particle distribution function.

Figure 2 also shows the diagrammatic expansion for the reduced three- and four-particle correlation functions. In the equation for the n-particle correlation function, the simplification of the diagrams with $n+1$ nodes was discussed above. We now discuss the simplification for the diagrams with n nodes for equations for the three-particle correlation function and higher correlation functions. The diagrams with $\leqslant(n-1)$ nodes are also present in the equations for correlation functions for $\leqslant(n-1)$ particles, and so these terms cancel

when the reduced equation is evaluated. The diagrams with n nodes on the right can be simplified as follows.

(i) Any diagram with one open circle not connected by a collision link can be factored into F^{CE} and a term in the equation for the lower-order correlation function. Therefore, these terms do not appear in the reduced diagram for the n-particle distribution function.

(ii) Any diagram for n particles with two open circles connected by a collision link is zero, since these can be factored into the products of the Boltzmann equation and the $(n-2)$-particle correlation function.

With the above two rules, it can be verified that there are only two diagrams with n nodes, one of which has n solid circles and the second of which has $n-1$ solid circles and the open circle with a collision link. The reduced diagrams for the three- and four-particle correlation functions are shown in Figs. 2(b) and 2(c).

We can now examine the most divergent of the higher-order correlation functions. We first assume that the solution of the two-particle (ring-kinetic) equation is valid and then show that the contribution to the higher-order correlation functions due to the most divergent part of the ring-kinetic equation is zero. To do this, we examine the left- and right-hand sides of the reduced equation for the three-particle correlation function, Fig. 2(b):

$$
\begin{aligned}
&F_\alpha^{CE}F_\beta^{CE}F_\gamma^{CE}(\partial_t + L_\alpha + L_\beta + L_\gamma)g_{\alpha\beta\gamma} \\
&\quad = \bar{\rho}^{-1}\delta(\mathbf{x}_{\alpha\beta})C_{\alpha\beta}[F_\alpha^{CE}F_\beta^{CE}F_\gamma^{CE}(g_{\alpha\gamma}+g_{\beta\gamma}+g_{\alpha\beta\gamma})] \\
&\qquad + \bar{\rho}^{-1}\delta(\mathbf{x}_{\alpha\gamma})C_{\alpha\gamma}[F_\alpha^{CE}F_\beta^{CE}F_\gamma^{CE}(g_{\alpha\beta}+g_{\beta\gamma}+g_{\alpha\beta\gamma})] \\
&\qquad + \bar{\rho}^{-1}\delta(\mathbf{x}_{\beta\gamma})C_{\beta\gamma}[F_\alpha^{CE}F_\beta^{CE}F_\gamma^{CE}(g_{\alpha\beta}+g_{\alpha\gamma}+g_{\alpha\beta\gamma})] \\
&\qquad + \int_\xi (C_{\alpha\xi}+C_{\beta\xi}+C_{\gamma\xi}) \\
&\qquad \times [F_\alpha^{CE}F_\beta^{CE}F_\gamma^{CE}(g_{\alpha\beta\gamma}+g_{\alpha\beta\xi}+g_{\beta\gamma\xi}+g_{\alpha\gamma\xi})]. \qquad (114)
\end{aligned}
$$

In the above equation, the inhomogeneous terms are the two-particle correlations on the right side.

It is easy to show that these inhomogeneous terms are zero in the leading approximation for the solutions of the ring-kinetic equations. First, note that the solutions of the ring-kinetic equation are of the type $(g_{\alpha\gamma}+g_{\beta\gamma})=A_{PQ}(\xi_{\alpha P}+\xi_{\beta P})\xi_{\gamma Q}$, where $\xi_{\alpha P}$ and $\xi_{\beta Q}$ are linear combinations of the variables conserved in a collision. Therefore, in a term of the type $C_{\alpha\beta}[F_\alpha^{CE}F_\beta^{CE}F_\gamma^{CE}(g_{\alpha\gamma}+g_{\beta\gamma})]$, $\xi_{\gamma Q}$ involves a particle index which is not undergoing a collision and $(\xi_{\alpha P}+\xi_{\beta P})$ is unchanged in a collision. Consequently, $C_{\alpha\beta}[F_\alpha^{CE}F_\beta^{CE}F_\gamma^{CE}(g_{\alpha\gamma}+g_{\beta\gamma})]$ is zero if F_α^{CE} and F_β^{CE} are Maxwell-Boltzmann distributions. This term is not zero, however, if the distribution function is not a Maxwell-Boltzmann distribution. However, in the limit $\epsilon\to 0$, the correction to the Maxwell-Boltzmann distribution is $O(\epsilon)$, and therefore, the term $C_{\alpha\beta}[F_\alpha^{CE}F_\beta^{CE}F_\gamma^{CE}(g_{\alpha\gamma}+g_{\beta\gamma})]$ is $O(\epsilon)$ smaller than $g_{\alpha\gamma}$. Since the inhomogeneous term in the equation for the three-particle correlation function in (114) is $O(\epsilon)$ smaller than $g_{\alpha\gamma}$, the three-particle correlation function $g_{\alpha\beta\gamma}$ is also $O(\epsilon)$ smaller than the two-particle distribution function $g_{\alpha\beta}$.

A similar procedure can be used for the four-particle and higher correlation functions. The inhomogeneous terms in the equation for the four-particle correlation function in Fig. 2 have collision terms with one solid and one open circle. These terms are also of the form $C_{\alpha\beta}[F_\alpha^{CE}(g_{\alpha\gamma\xi}+g_{\beta\gamma\xi})]$. If the three-particle correlations are in the form of products of collisional invariants, then the terms of the form $C_{\alpha\beta}[F_\alpha^{CE}(g_{\alpha\gamma\xi}+g_{\beta\gamma\xi})]$ are also zero if the single-particle distribution functions are Maxwell-Boltzmann distributions. Due to the modification of the Maxwell-Boltzmann distribution due to the mean shear, there are corrections to these terms of $O(\epsilon)$; due to these corrections, the four-particle correlation function is $O(\epsilon)$ smaller than the three-particle correlation function. The same inference can be drawn for the higher-order correlation functions as well. This indicates that the two-particle correlation function from the ring-kinetic equation provides the most singular contribution to the transport coefficients in the limit $\epsilon\to 0$ (or $\dot{\gamma}\to 0$).

VII. CONCLUSIONS AND FUTURE DIRECTIONS

The ring-kinetic equation for the two-particle distribution function was analyzed in Sec. II. The equation for the two-particle correlation function has the form $(\partial_t+\tilde{L}_{\mathbf{k}}^{\alpha}+\tilde{L}_{-\mathbf{k}}^{\beta})\tilde{g}_{\alpha\beta}=\mathcal{R}$, where the inhomogeneous term $\mathcal{R}$ is related to the first term in the Chapman-Enskog solution for the Boltzmann equation. The critical piece of insight used here was that the largest contribution to the two-particle correlation function results from products of the conserved eigenvectors of the operators $\tilde{L}_{\mathbf{k}}^{\alpha}$ and $\tilde{L}_{-\mathbf{k}}^{\beta}$, which have zero eigenvalues in the limit $\mathbf{k}\to 0$. These are identical to the eigenvectors for the linearized Boltzmann or linearized Navier-Stokes equations. When this two-particle correlation function is substituted into the ring-kinetic equation, we obtain a first-order differential equation in time for the two-particle correlation function, which also contains a derivative with respect to wave vector due to the linear shear flow. Ernst *et al.* [11] solved this equation, at steady state, by integrating in the wave vector coordinate. Here, we use a time-dependent wave vector to get rid of the derivative with respect to wave vector and carry out an explicit time integration in order to determine the two-particle correlation function and the resulting correction to the single-particle correlation function. From this, the stress is calculated by taking the appropriate second moment of the fluctuating velocity.

Ernst *et al.* [11] used the ring-kinetic equation to calculate the singular contribution to the stress in the limit $\dot{\gamma}\to 0$. In our calculation, the strain rate $\dot{\gamma}$ and the temperature are related through the energy balance equation. In the elastic limit $e\to 1$, the strain rate $\dot{\gamma}\sim\epsilon(\sqrt{T}/\lambda)$, where T is the temperature and λ is the mean free path. Therefore, the expansion for a sheared inelastic steady state is in the parameter ϵ, which provides the extent of inelasticity in the system. In the time-integration scheme followed here, we show how the higher-order terms can be evaluated; these turn out to be functions of the higher-order Chapman-Enskog solutions for the Boltzmann equation. In fact, the series can be explicitly resummed, to obtain an equation (76), which is not valid to

all orders in ϵ, provided the solutions of the Boltzmann equation are known from the Chapman-Enskog procedure.

It was shown, in Sec. II, that the leading order solution for the viscosity is identical to the Green-Kubo formula, provided the *velocity autocorrelation functions are calculated in the sheared state*. Note that a definite phase-space distribution in the equilibrium (reference) state is assumed while calculating the Green-Kubo formula, whereas we have made no assumption about the phase-space distribution in the sheared state. This is similar to the derivation of the Maxwell-Boltzmann distribution from the Boltzmann distribution, where the equal probability of microstates in the Gibbs ensemble is not assumed in the calculation. The solution of the Boltzmann equation can be extended to systems near equilibrium using the Chapman-Enskog procedure, which involves an expansion in the ratio of the mean free path and the macroscopic scale. In a similar manner, we have shown how the most singular solution of the ring-kinetic equation can be extended to higher orders in the parameter ϵ. We have also been able to explicitly resum the series to obtain Eq. (76). The term $(\mathcal{I}_{IJ}^{MN}-\Theta_{IJ}^{KL}K_{KL}^{MN})^{-1}$ on the right-hand side of Eq. (76) also has the potential to provide an specific interpretation of "jamming" or "slow dynamics" as ϵ increases. This is because a solution for the stress exists only if the matrix $(\mathcal{I}_{IJ}^{MN}-\Theta_{IJ}^{KL}K_{KL}^{MN})$ is invertible. If the determinant of the matrix becomes zero for some value of ϵ, it implies that the stress is divergent even at finite strain rate, for which the Chapman-Enskog solution for the single-particle distribution function is finite. The matrix $(\mathcal{I}_{IJ}^{MN}-\Theta_{IJ}^{KL}K_{KL}^{MN})^{-1}$ depends only on the hydrodynamic modes of the linearized Boltzmann equation for a shear flow and is independent of the strength of the corrections $h_{\alpha}^{(1)},h_{\alpha}^{(2)},\ldots$ for the Chapman-Enskog solution for the velocity distribution function. In addition, the matrices Θ_{IJ}^{KL} and K_{KL}^{MN} are functions of the type of forcing through the operator S_{α}' in Eqs. (61) and (71). A steady shear flow has been used as the forcing in the present case, but we have indicated how the analysis could be extended to other types of forcing. It is clear that if the matrix $(\mathcal{I}_{IJ}^{MN}-\Theta_{IJ}^{KL}K_{KL}^{MN})$ has a zero determinant, the divergence of the stress response is specific to the type of forcing used. Therefore, this represents a jamming or slowing down of the sheared steady state, which is a nonequilibrium transition. There is the scope for much further research on the nature of this transition.

The results of Sec. II were used to calculate the stress response of an inelastic sheared fluid. For an elastic fluid, the results of Ernst *et al.* [11] were recovered by this procedure. For an inelastic fluid, we have explicitly calculated the renormalization of the viscosity. Due to the $k^{2/3}$ dependence of the growth rate of the hydrodynamic modes, we find that the correction to the viscosity due to correlations is $O(\epsilon^{5/2})$ in two dimensions [in contrast to the $\ln(\epsilon)$ dependence for an elastic fluid] and is $O(\epsilon^{15/4})$ in three dimensions (in contrast to the $\epsilon^{1/2}$ dependence for an elastic fluid). Therefore, the viscosity is not divergent in two dimensions in an inelastic fluid and the Burnett coefficients are not divergent in three dimensions if the length scale is large compared to the conduction length. This indicates that the difficulties associated with the long-time tails of the velocity autocorrelation function are not present in inelastic fluids.

It should be noted that the distinction between elastic and inelastic fluids used here is rather artificial, and the real distinction is whether the length scale is larger or smaller than the conduction length (λ/ϵ), where λ is the mean free path and $\epsilon=(1-e)^{1/2}$. It is necessary to devise techniques to treat these two wave number ranges within the framework or to be able to use a matched asymptotic expansion for a smooth transition from one wave number range to the other in order to be able to obtain quantitative predictions. Though the conduction length is large compared to the microscopic scale for $\epsilon \ll 1$, this distinction will not be clearly defined for real granular materials for which e is not close to 1. Therefore, in real applications, there will be very little distinction between the microscopic scale and the conduction length, and it is appropriate to use the eigenvalues and eigenfunctions obtained for an inelastic fluid for calculating the transport coefficients.

The three-particle and higher correlation functions were analyzed in Sec. III. An agreement between the Green-Kubo formula and the leading-order solution for the viscosity from the ring-kinetic equation implies that the higher-order correlations may not result in a dominant contribution to the viscosity. In Sec. III, we show explicitly that when the two-particle correlation function is in the form of the product of the eigenfunctions for the conserved hydrodynamic modes for the two particles, the three-particle and higher-order correlation functions are small compared to the two-particle correlation function. Thus, the two-particle correlation function obtained from the ring-kinetic equation provides the largest contribution to the transport coefficients in the limit $\epsilon \to 0$.

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